Numbers

- Natural numbers ($\mathbb{N}$): 0,1,2, ...
- Integer numbers ($\mathbb{Z}$): Naturals + negatives (-1,-2,-3, ...)
- Rational numbers ($\mathbb{Q}$): Integers + fractions ($\frac{1}{2}$, $-\frac{7}{5}$, ...)
- Real numbers ($\mathbb{R}$): Rationals + other (e.g. $\sqrt{3}$, π, ...)
- Complex numbers ($\mathbb{C}$): Reals + imaginary (...)

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \dots$$

Functions

$$\boxed{\text{Input}} \xrightarrow{\text{Rule}} \boxed{\text{Output}}$$

- Notation:

$$\text{Name} \colon \text{Domain} \rightarrow \text{Range}$$

- Terminology:

Function name defined on domain with values in range

Function Cos defined on $\mathbb{R}$ with values in $\mathbb{R}$

Function Cos defined on $\mathbb{R}$ with values in $[-1, 1]$

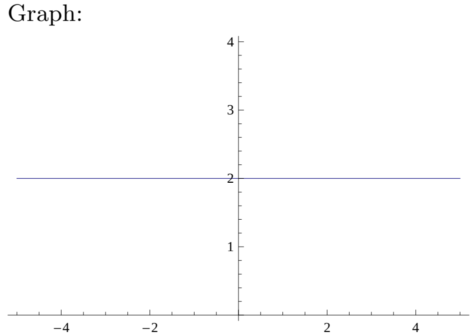
Function Cos defined on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with values in $[-1, 1]$

- Numerical Functions

$$f \colon \mathbb{R} \rightarrow \mathbb{R} \quad \text{OR} \quad f \colon A \rightarrow \mathbb{R} \text{ with } A \subset \mathbb{R}$$

Constant functions

- General form: $f \colon \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = c$
- Example: $f(x) = 2$ (sometimes denoted by $f(x) \equiv 2$)
- Graph:



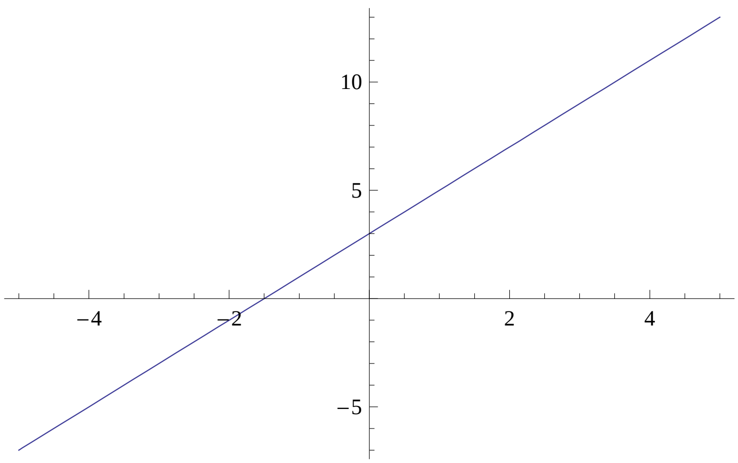
- Important, but boring ...

Linear functions

- General form: $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = ax$
  - Examples: $f(x) = 2x$, $f(x) = -\pi x$
- a: slope
- Properties:

$$f(x+y) = f(x) + f(y)$$
$$f(kx) = kf(x)$$

  - The only functions for which the above rule works!
- Linear for calculus: $f(x) = ax + b$
  - Example: $f(x) = 2x - 4$, $f(x) = 3(x-1) + 5$
  - Do not satisfy the above rules unless $b = 0$!!
- a: slope, b: intercept



Power functions

- General form: $f\colon M \subset \mathbb{R} \to \mathbb{R}$, $f(x) = x^a$
  - Examples: $f(x) = x^3$, $f(x) = x^{-1}$, $f(x) = x^{1/2}$
  - Variable base, constant exponent
  - Domain is not always all $\mathbb{R}$!
  - If p is integer and q is natural

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

  - What does x^π really mean?
- Power Rules
  - $x^m \cdot x^n = x^{m+n}$
  - $\frac{x^m}{x^n} = x^{m-n}$
  - $(x^m)^n = x^{m \cdot n}$

Polynomial functions

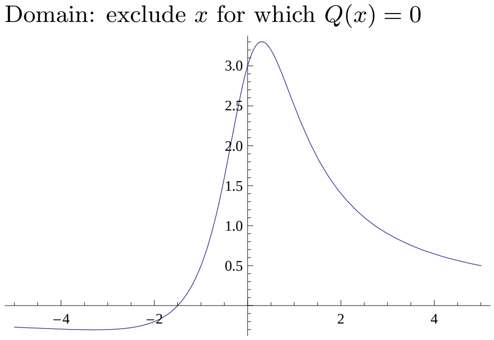
- Combination of powers with natural exponents

- Exponent of highest power: degree

- Examples
  - $f(x) = -x^3 + 4\pi x$, degree 3 $\to$ cubic polynomial
  - $f(x) = ax^2 + bx + c$, general form of quadratic
  - $f(x) = ax + b$, linear (for calculus)
  - Domain is all $\mathbb{R}$!
  - Long term behavior (larger and larger, or smaller and smaller values of x)? depends on degrees and leading coefficient
  - Non-example: $f(x) = 4x^3 + x^{-2}$

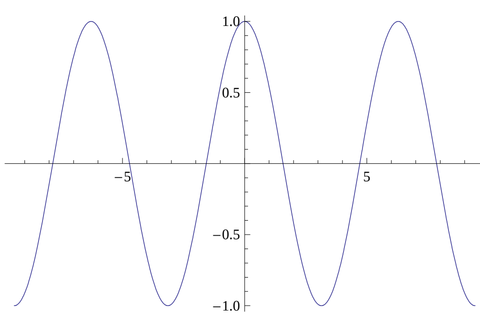
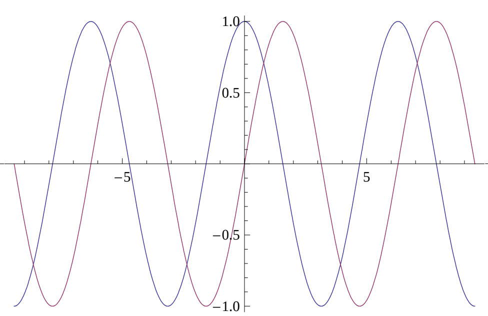
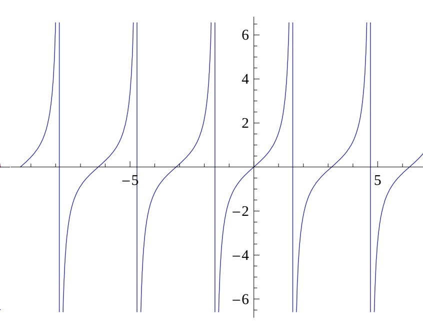
Rational functions

- Ratios of polynomial functions
- $f(x) = \frac{P(x)}{Q(x)}$
- Examples
  - $f(x) = 4x^3 + x^{-2} = \frac{4x^5+1}{x^2}$
  - $f(x) = \frac{x^3-\pi x}{x^2+1}$
- Domain: exclude x for which $Q(x) = 0$



- Long term behavior depends on degrees and leading coefficients

Trigonometric functions

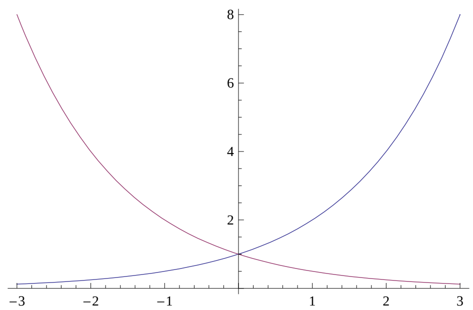
- sin, cos, $\tan = \frac{\sin}{\cos}$, $\sec = \frac{1}{\cos}$
- Domains
  - sin, cos: all $\mathbb{R}$
  - tan, sec: exclude x for which $\cos x = 0$ $\quad x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$
-   

- Periodicity $f(x+T) \equiv f(x)$ for all x.
  - sin, cos, sec: period 2π
  - tan: period π
- Fundamental formula:

$$\sin^2 x + \cos^2 x \equiv 1$$

Exponential functions

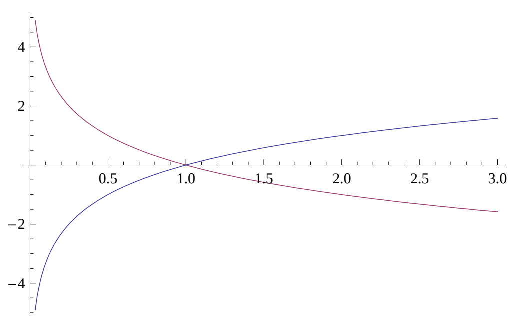
- General form: $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$
- Defined everywhere only for $a > 0$
- Constant base, variable exponent
- Examples: $f(x) = 2^x$, $f(x) = \left(\frac{1}{2}\right)^x$
- DO NOT CONFUSE with power functions!! Constant exponent, variable base



- Long term behavior
- What does π^x really mean?

Logarithmic functions

- To what power should we raise 2 to get 16?
- $\log_2 16 = 4$
- Logarithmic function: $f\colon (0, \infty) \to \mathbb{R}$, $f(x) = \log_a x$
  - To what power should we raise a to get x?
  - Only makes sense if $a > 0$ and $a \neq 1$



- Exponentials and logarithms

$$a^{\log_a x} = x \qquad \log_a(a^x) = x$$