Operations with Functions

Math 140 - Calculus I

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September 10, 2009

Algebraic Operations

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$A \subseteq \mathbb{R}$, subset of real numbers
$f, g\colon A \to \mathbb{R}$, two numerical functions

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  Defined only on $B = A \setminus \{x \mid g(x) = 0\}$

Scaling and Shifting

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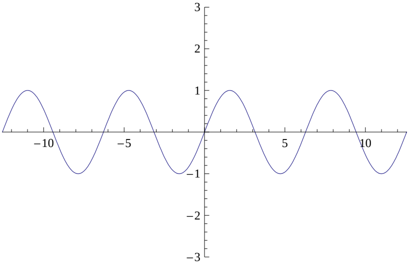
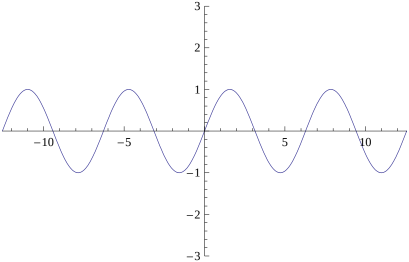


$$t \to \sin(t)$$

Scaling and Shifting

$$f(t) = A\sin(\omega t + \phi) \qquad\qquad A = 1, \quad \omega = 3, \quad \phi = 0$$



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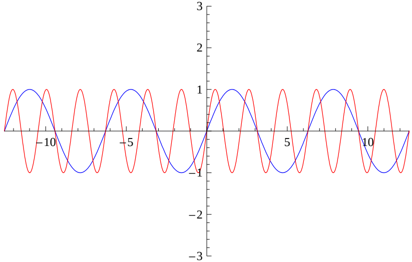

$$t \to \sin(t) \to t \to \sin(3t)$$

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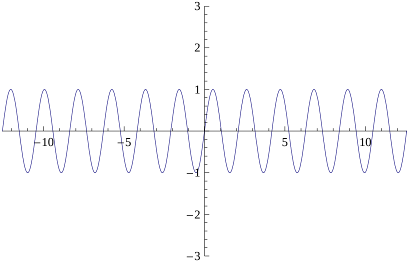

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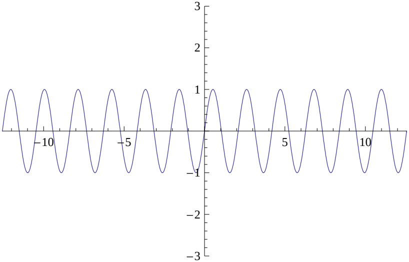

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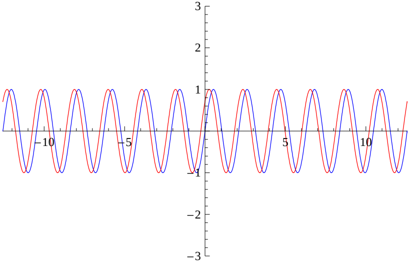

$t \to \sin(3t) \to t \to \sin(3t + \pi/4)$

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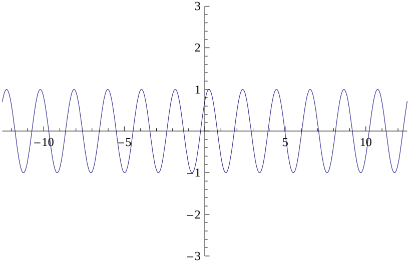

$$t \to \sin(3t + \pi/4)$$

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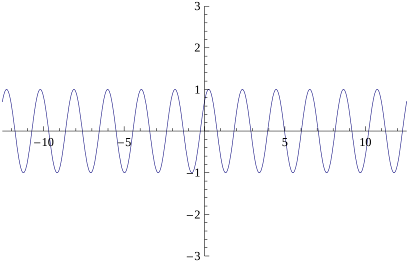
$$f(t) = A\sin(\omega t + \phi) \qquad\qquad A = 2, \quad \omega = 3, \quad \phi = \pi/4$$



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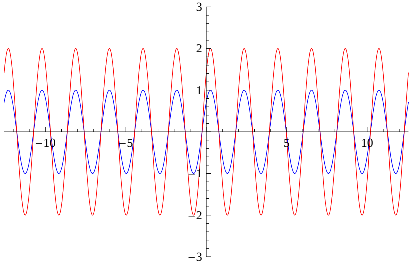

$$t \to \sin(3t + \pi/4) \to t \to 2\sin(3t + \pi/4)$$

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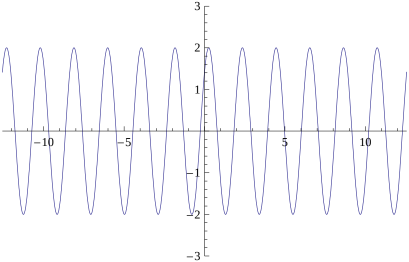

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A ball is thrown vertically up from a height of 1m, with an initial velocity of 10m/s.

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- Parameters

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Completing the square

$$\begin{aligned} h(t) &= 1 + 10t - 5t^2 \\ &= -5t^2 + 10t + 1 = \\ &= -5(t^2 - 2t) + 1 = \\ &= -5(t^2 - 2t + 1 - 1) + 1 = \\ &= -5((t-1)^2 - 1) + 1 = \\ &= -5(t-1)^2 + 6 \end{aligned}$$

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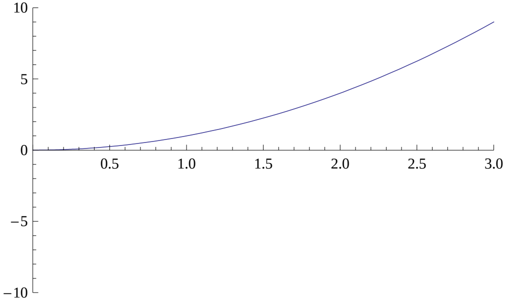

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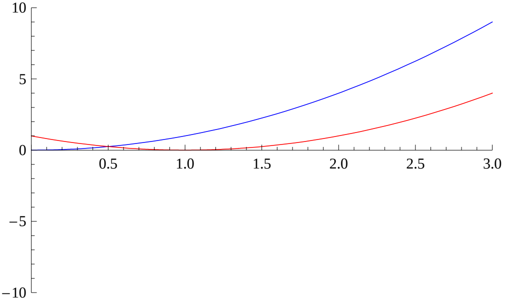

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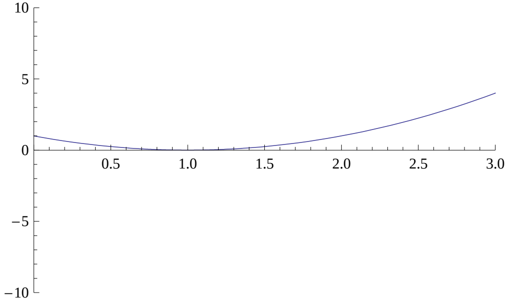

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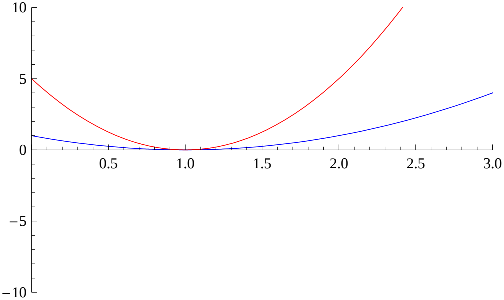

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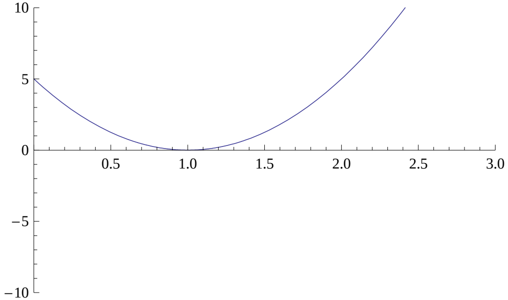

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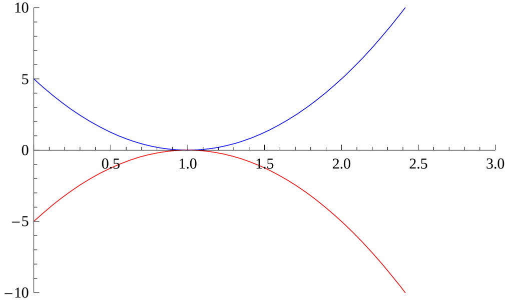

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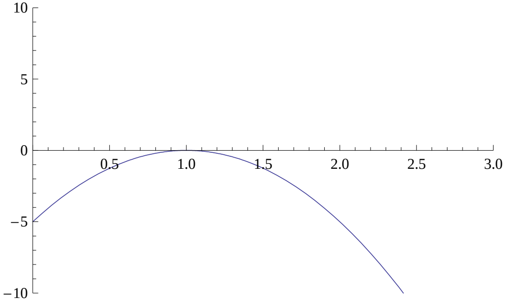

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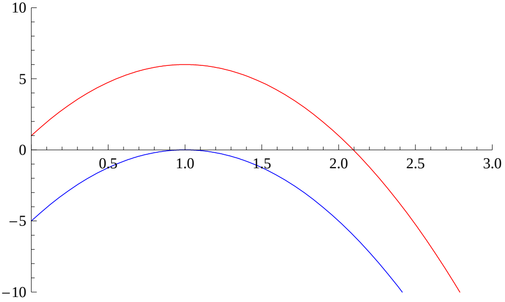

$$t \to -5(t-1)^2 \Longrightarrow t \to 6 - 5(t-1)^2$$

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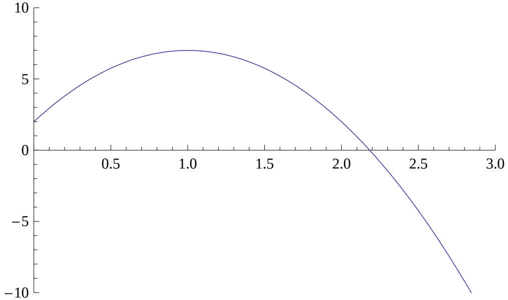

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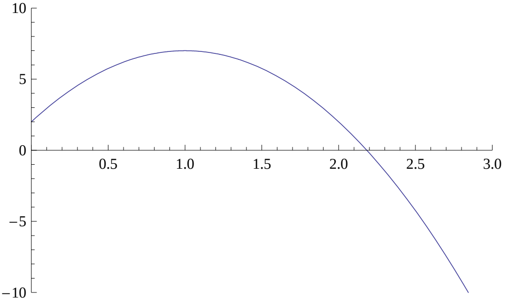

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Domain?

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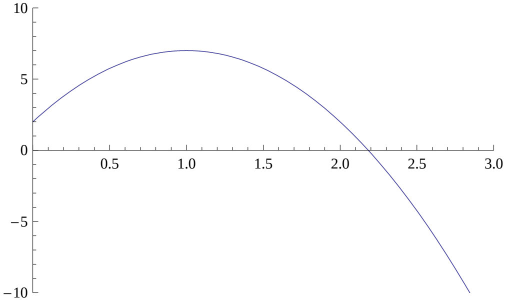

$$t \to -5(t-1)^2 + 6$$

Graph vs. Trajectory

Composition of Functions

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- Rule f: subtract one unit from the input

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  Input for h: x

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  Input for h: $x \Longrightarrow$ Input for f: x

$$\boxed{x} \xrightarrow{f} \boxed{x} - 1 = x - 1$$

Composition of Functions

- Rule f: subtract one unit from the input
$$\square \xrightarrow{f} \square - 1 \iff f(\square) = \square - 1$$
- Rule g: square the input
$$\square \xrightarrow{g} \square^2 \iff g(\square) = \square^2$$
- Rule h: First apply rule f, then apply rule g
Input for h: $x \Longrightarrow$ Input for f: x
$$\boxed{x} \xrightarrow{f} \boxed{x} - 1 = x - 1$$
Output of f: $x - 1 \Longrightarrow$ Input for g: $x - 1$

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- Composition $g \circ f$ only makes sense when
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  Range of f is included in domain of g

Decomposition of functions

$$h(x) = \cos^3(2x) = (\cos(2x))^3$$

Write as a composition of simpler rules

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- $x = \pi/4 \Longrightarrow 2x = 2 \cdot \pi/4 = \pi/2 \Longrightarrow$ Double the input:

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Last operation performed: p

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First operation performed: $f \Longrightarrow \triangle = f(x) = 2x$

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$$h(x) = p(\square) = p(g(\triangle)) = p(g(f(x)) \Longleftrightarrow h = p \circ g \circ f$$