Algebraic Operations

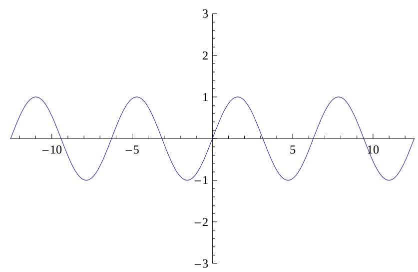
$A \subseteq \mathbb{R}$, subset of real numbers

$f, g\colon A \to \mathbb{R}$, two numerical functions

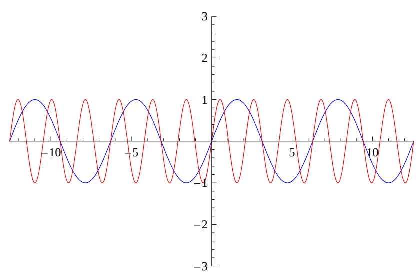
- Sum of functions $h = f + g$, $h\colon A \to \mathbb{R}$, $h(x) = f(x) + g(x)$
- Difference of functions $h = f - g$, $h\colon A \to \mathbb{R}$, $h(x) = f(x) - g(x)$
- Product of functions $h = fg = f \cdot g$, $h\colon A \to \mathbb{R}$, $h(x) = f(x) \cdot g(x)$
- Ratio of functions $h = f/g$, $h\colon B \to \mathbb{R}$, $h(x) = f(x)/g(x)$ Defined only on $B = A \setminus \{x \mid g(x) = 0\}$

Scaling and Shifting

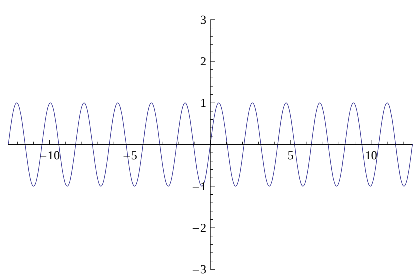
$$f(t) = A\sin(\omega t + \phi)\;\; f(t) = A\sin(\omega t + \phi)\;\; f(t) = A\sin(\omega t + \phi)\;\; f(t) = A\sin(\omega t + \phi)$$
$$A = 1, \quad \omega = 1, \quad \phi = 0\;\; A = 1, \quad \omega = 3, \quad \phi = 0\;\; A = 1, \quad \omega = 3, \quad \phi = 0\;\; A = 1, \quad \omega = 3, \quad \phi = \pi/4$$
$$A = 1, \quad \omega = 3, \quad \phi = \pi/4\;\; A = 2, \quad \omega = 3, \quad \phi = \pi/4\;\; A = 2, \quad \omega = 3, \quad \phi = \pi/4$$



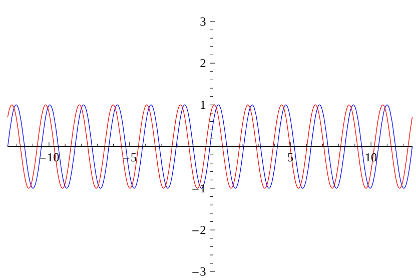
$t \to \sin(t)$



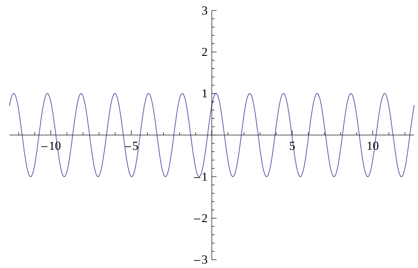
$t \to \sin(t) \to t \to \sin(3t)$



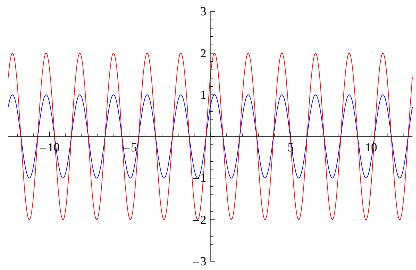
$t \to \sin(3t)$



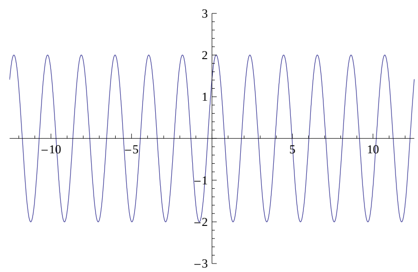
$t \to \sin(3t) \to t \to \sin(3t + \pi/4)$



$t \longrightarrow \sin(3t + \pi/4)$



$t \longrightarrow \sin(3t + \pi/4) \longrightarrow t \longrightarrow 2\sin(3t + \pi/4)$



$t \longrightarrow 2\sin(3t + \pi/4)$

Another Example

A ball is thrown vertically up from a height of 1m, with an initial velocity of 10m/s. Let $h(t)$ be the height of the ball after t seconds, measured from the ground.

- Dependent variable: height, h (in meters)
- Independent variable: time, t (in seconds)
- Rule of assignment? Law of Physics

$$h(t) = h_0 + v_0 t - \frac{1}{2} g t^2$$

- Parameters
  - h_0 = initial height = 1m
  - v_0 = initial velocity = 10m/s
  - g = gravitational acceleration $\simeq 10 m/s^2$

$$h(t) = 1 + 10t - 5t^2$$

Graph of h

$$h(t) = 1 + 10t - 5t^2$$

Graph: parabola $\Longleftarrow$ quadratic function

From $f(t) = t^2$ to $h(t) = 1 + 10t - 5t^2$

Completing the square

$$\begin{aligned} h(t) &= 1 + 10t - 5t^2 \\ &= -5t^2 + 10t + 1 = \\ &= -5(t^2 - 2t) + 1 = \\ &= -5(t^2 - 2t + 1 - 1) + 1 = \\ &= -5((t-1)^2 - 1) + 1 = \\ &= -5(t-1)^2 + 6 \end{aligned}$$

$$h(t) = 1 + 10t - 5t^2 = -5(t-1)^2 + 6$$
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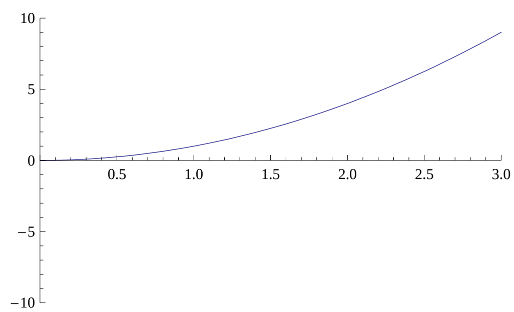
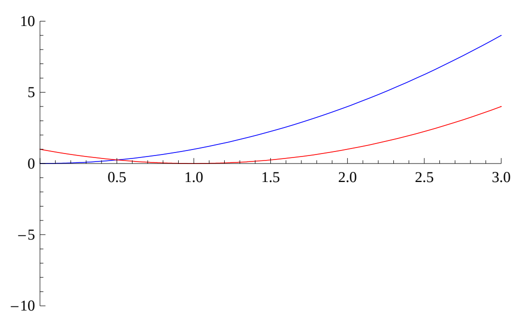
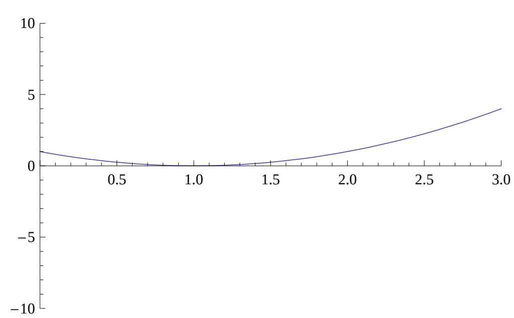


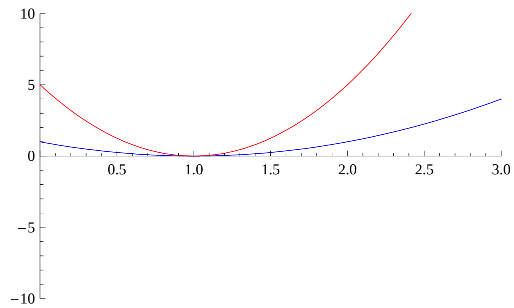
$$t \to t^2$$



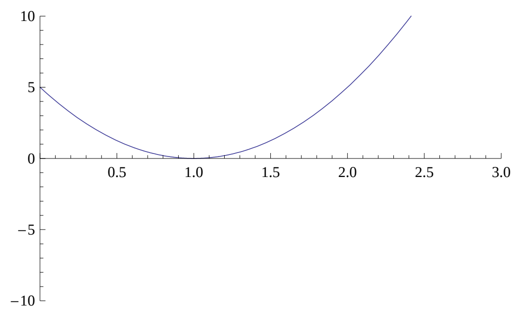
$$t \to t^2 \Longrightarrow t \to (t-1)^2$$



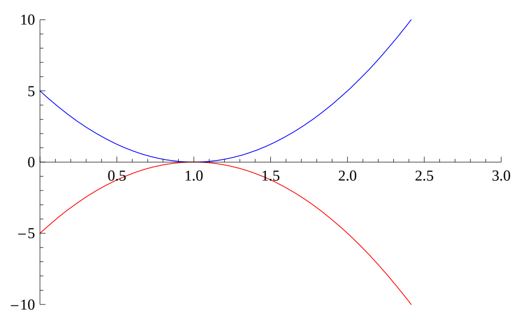
$$t \to (t-1)^2$$



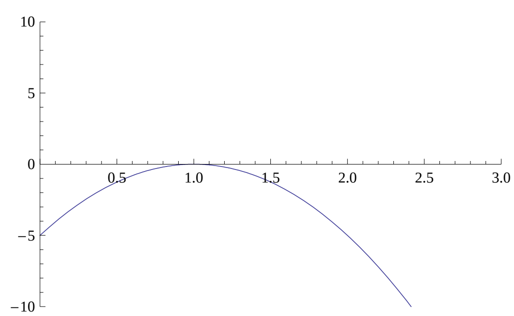
$$t \to (t-1)^2 \Longrightarrow t \to 5(t-1)^2$$



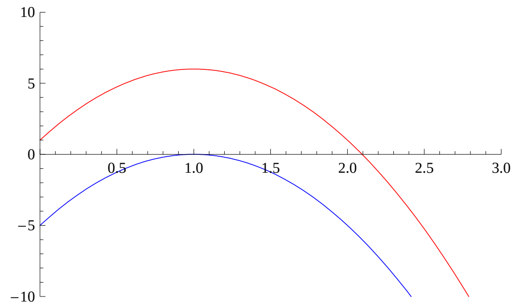
$$t \to 5(t-1)^2$$



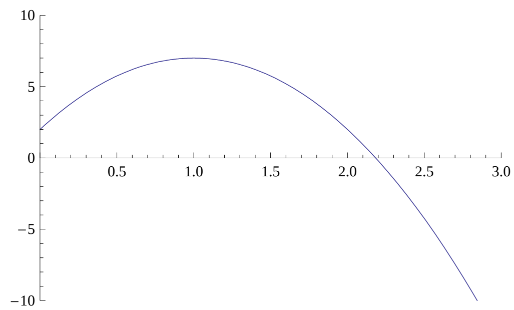
$$t \to 5(t-1)^2 \Longrightarrow t \to -5(t-1)^2$$



$$t \to -5(t-1)^2$$



$$\color{blue}{t \to -5(t-1)^2} \Longrightarrow \color{red}{t \to 6-5(t-1)^2}$$



$$t \to -5(t-1)^2 + 6$$

Domain?

Graph vs. Trajectory

Composition of Functions

- Rule f: subtract one unit from the input

$$\square \xrightarrow{f} \square - 1 \Longleftrightarrow f(\square) = \square - 1$$

- Rule g: square the input

$$\square \xrightarrow{g} \square^2 \Longleftrightarrow g(\square) = \square^2$$

- Rule h: First apply rule f, then apply rule g Input for h: $x \Longrightarrow$ Input for f: x

$$\boxed{x} \xrightarrow{f} \boxed{x} - 1 = x - 1$$

Output of f: $x - 1 \Longrightarrow$ Input for g: $x - 1$

$$\boxed{x-1} \xrightarrow{g} \boxed{x-1}^2 = (x-1)^2$$

$$x \ \xrightarrow{f} \ x{-}1 \ \xrightarrow{g} \ (x{-}1)^2 \Longrightarrow h(x) = (x-1)^2$$

$$x \ \xrightarrow{f} \ x{-}1 \ \xrightarrow{g} \ (x{-}1)^2 \Longrightarrow h(x) = (x-1)^2$$

- $h = g \circ f$: composition of g and f $\qquad h(x) = g(f(x))$
- First function applied = innermost = last mentioned
- Order matters:

$$f(g(x)) = x^2 - 1 \neq (x-1)^2 = g(f(x))$$

- Composition $g \circ f$ only makes sense when Potential outputs of f are valid inputs for g Range of f is included in domain of g

Decomposition of functions

$$h(x) = \cos^3(2x) = (\cos(2x))^3$$

Write as a composition of simpler rules

Strategy: What operations are performed?

Example: To compute $h(\pi/4)$:

- $x = \pi/4 \Longrightarrow 2x = 2 \cdot \pi/4 = \pi/2 \Longrightarrow$ Double the input: $f(\square) = 2\square$
- $2x = \pi/2 \Longrightarrow \cos(2x) = \cos(\pi/2) = 0 \Longrightarrow g(\square) = \cos(\square)$
- $\cos(2x) = 0 \Longrightarrow (\cos(2x))^3 = 0^3 = 0 \Longrightarrow$ Cube: $p(\square) = \square^3$

Last operation performed: p

$$h(x) = \square^3 = p(\square)$$

Prior to that: $g \Longrightarrow \square = \cos(\triangle)$

$$h(x) = p(\square) = p(g(\triangle))$$

First operation performed: $f \Longrightarrow \triangle = f(x) = 2x$

$$h(x) = p(\square) = p(g(\triangle)) = p(g(f(x)) \Longleftrightarrow h = p \circ g \circ f$$