# Rectangular Coordinates

January 27, 2010

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Relationships:

- Inclusion - belongs to, is on
- Parallelism
  - Parallel planes or line parallel to a plane:
    empty intersection
  - Parallel lines:
    included in a common plane, but empty intersection
  - Skew lines:
    not included in a common plane

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- Measures of angles $\longrightarrow$ Perpendicularity
  - Line and Line
  - Line and Plane
  - Plane and Plane

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- Rectangular Coordinate System
  - Position $\rightarrow$ Origin $\rightarrow$ $O$;
  - Orientation $\rightarrow$ Fundamental directions $\rightarrow$ Ahead-Left-Up;
  - Displacement $\rightarrow$ measured using the distance:
    - Positive in Ahead, Left, and Up directions;
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- Example: Projector $P \rightarrow (x, y, z) = (3, 1, 2)$ or $P(3, 1, 2)$.
- Using a fixed coordinate system:
  - Space $\simeq R \times R \times R = R^3$
  - Plane $\simeq R \times R = R^2$

# Euclidean Distance in Coordinates

Points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$:

$$d(A, B) = |AB| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

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Example: $P(3,1,2)$ and $Q(1,2,3)$:

$$D(P,Q) = \sqrt{(1-3)^2 + (2-1)^2 + (3-2)^2} = \sqrt{6}\,.$$

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General formula for this change of coordinates:

$$x' = -y \quad (1)$$
$$y' = x - 1 \quad (2)$$
$$z' = z \quad (3)$$

Example $Q(x = 1, y = 2, z = 3) \leftrightarrow Q(x' = -2, y' = 0, z' = 3)$

# Fundamental Philosophy

$P(3, 1, 2)$ and $Q(1, 2, 3)$ in $(x, y, z)-$coordinates
$P(-1, 2, 2)$ and $Q(-2, 0, 3)$ in $(x', y'z')-$coordinates
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Fundamental Philosophy:

- Space doesn't come with coordinates
- Natural concepts (such of distance) are independent of coordinates
- If a natural concept is defined using coordinates,
  the result does not depend on the chosen coordinate system.

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$X$ subset of a set $Y$:

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Important: Equations in Plane vs. Space.

# Recognizing Spheres from Equations

$Q(x_0, y_0, z_0)$, $r > 0$, $A(x, y, z)$. Remark: $d(A, Q) = r \longleftrightarrow d^2(A, Q) = r^2$

$$S_r(Q): \quad (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

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$$x^2 + y^2 + z^2 - 4x + 2z - 4 = 0$$

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Examples:

$$x^2 + y^2 + z^2 - 4x + 2y = 0$$

Complete the square:

$$(x - 2)^2 + (y + 1)^2 + z^2 = 5$$

Spere of radius $\sqrt{5}$ centered at $(2, -1, 0)$.

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How about $x^2 + y^2 + z^2 - 4x + 2y = 6$? Passes both tests, but ...

$$(x-2)^2 + (y+1)^2 + z^2 = -1$$

is impossible! No such points, set is empty.