Vectors

February 1, 2010

Motivation

- Location of projector from current position and orientation:
  - direction of projector
  - distance to projector
- (direction, distance=magnitude) $\Longleftrightarrow$ *vector*
- Examples:
  - Force
  - Velocity
  - Displacement

Displacement Vectors

Displacement vector = ordered pair of points, (A, B)

- Represented as an arrow
- A: tail, B: head
- Notation: $(A, B) = \overrightarrow{AB} = \mathbf{AB}$
- Magnitude: $|\mathbf{AB}| = |\overrightarrow{AB}|$.
- Direction: Geometric direction from A to B, if $A \neq B$
- If $A = B$:
  - Zero magnitude and non-specified direction
  - $(A, A) = \overrightarrow{AA} = \mathbf{AA}$: zero displacement vector.
- Displacement vector with tail fixed at O:
  - Position vector with respect to O;
  - $(O, P) = \overrightarrow{OP} = \mathbf{OP} = \mathbf{r}_P$.

Equality and Equivalence of Displacement Vectors

- Equality of displacement vectors:

$$\mathbf{AB} = \mathbf{DC} \iff (A, B) = (D, C) \iff A = D \text{ and } B = C$$

  - Useful for position vectors:

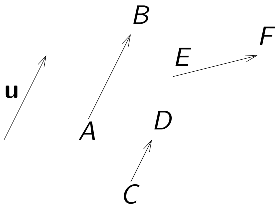
$$\mathbf{OP} = \mathbf{OQ} \iff P = Q\ .$$

  - In general too restrictive.
- Equal displacement vectors $\rightarrow$ same magnitude and direction.
- Same magnitude and direction $\not\rightarrow$ Equal displacement vectors.
- Equivalent displacement vectors $\Leftrightarrow$ Same magnitude and direction

$$\mathbf{AB} \equiv \mathbf{DC} \iff ABCD \text{ is a parallelogram}\ .$$

Vectors

- Vector $\mathbf{u}$:
  set of displacement vectors with given direction and magnitude
- Magnitude of $\mathbf{u}$: common given magnitude.
- Direction of $\mathbf{u}$: common given direction, if non-zero magnitude.
- Set of zero displacement vectors = zero vector, $\mathbf{0}$.
- Representative for $\mathbf{u}$:
  displacement vector $\mathbf{AB}$ with the same direction and magnitude



- Intuitive notation: $\mathbf{u} = \mathbf{AB}$.
- Graphical representation: arrow without fixed tail and head.
- Major advantage: we can translate displacement vectors.

Addition of Vectors

- By adding representative displacement vectors: Triangle Rule

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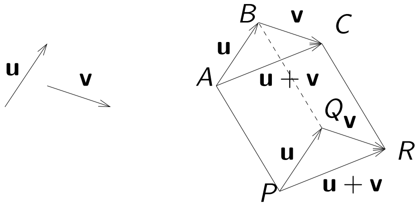

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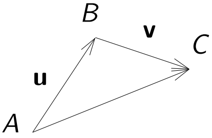

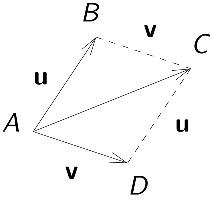

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- Properties:
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  - Associative, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
    Extends addition to $\mathbf{u} + \mathbf{v} + \mathbf{w}$

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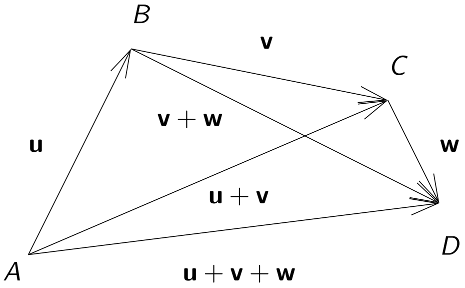

Figure: Sum of three vectors

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  - Opposite vector: If $\mathbf{u} = \mathbf{AB}$, then $\mathbf{AB} + \mathbf{BA} = \mathbf{0}$, hence $\mathbf{BA} = -\mathbf{u}$.

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- Difference of vectors: $\mathbf{u} - \mathbf{v} = -\mathbf{v} + \mathbf{u}$: Parallelogram rule.

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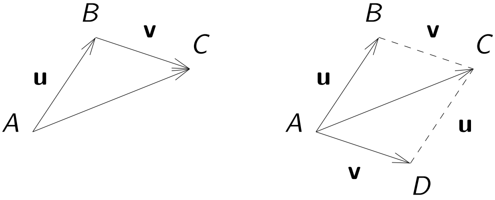

  - Associative, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
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Linear Combinations

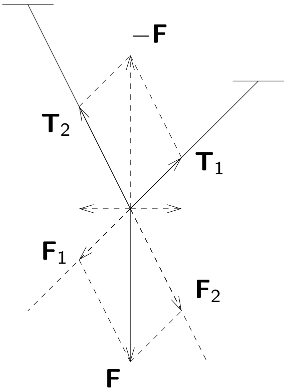
- Scalar multiples: Let $\mathbf{u}$ be a vector and c a real number (scalar)
  - If $c > 0$ then $c\mathbf{u}$ is the vector:
    - with the same direction
    - with magnitude $|c\mathbf{u}| = c|\mathbf{u}|$.
  - If $c < 0$ then $c\mathbf{u} = (-c)(-\mathbf{u})$:
    - opposite direction
    - magnitude $|c\mathbf{u}| = |(-c)(-\mathbf{u})| = (-c)| - \mathbf{u}| = |c||\mathbf{u}|$
  - If $c = 0$ then $c\mathbf{u} = \mathbf{0}$.
- If $c_1, \ldots, c_n$ are scalars and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are vectors, then

$$\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$$

  is the linear combination of given vectors with given scalars.

Decomposition of a vector along given directions

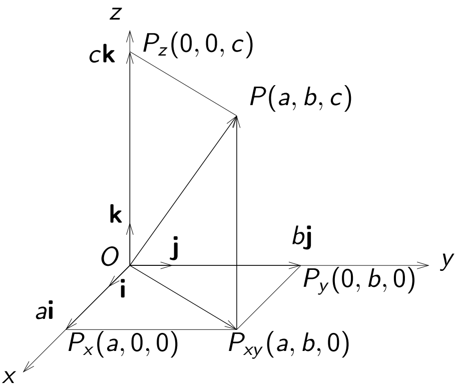
Example: Tension induced by given force.



Vectors in Coordinates

- *Oxyz*: fixed rectangular coordinate system
- **i**, **j**, **k**: unit vectors in the fundamental directions

If $P(a, b, c)$ is a point, then



$$\mathbf{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \langle a, b, c \rangle \ .$$

Operations in Coordinates

- Magnitude:
$$|\langle a, b, c\rangle| = |OP| = \sqrt{a^2 + b^2 + c^2}$$
- Addition:
$$\langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle \ .$$
- Scalar multiple:
$$c\langle x, y, z\rangle = \langle cx, cy, cz\rangle \ .$$
- General displacement from $A(x_A, y_A, z_A)$ to $B(x_B, y_B, z_B)$:
$$\mathbf{AB} = \mathbf{AO} + \mathbf{OB} = \mathbf{OB} - \mathbf{OA} = \langle x_B - x_A, y_B - y_A, z_B - z_A\rangle \ .$$