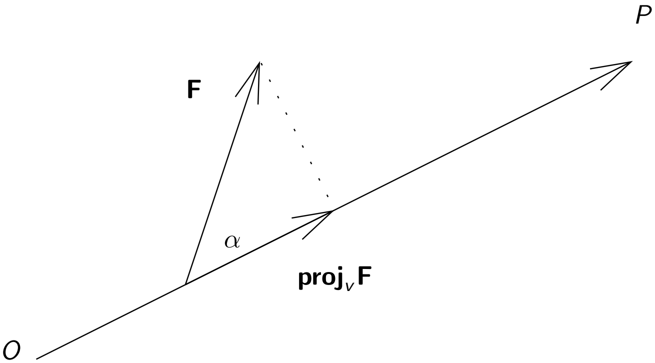
Dot Product

February 3, 2010

Motivation

Work done by a constant force



$$W = |\mathbf{proj}_v \mathbf{F}|\,|\mathbf{OP}| = |\mathbf{F}|\,|\mathbf{OP}|\,\cos\alpha\ .$$

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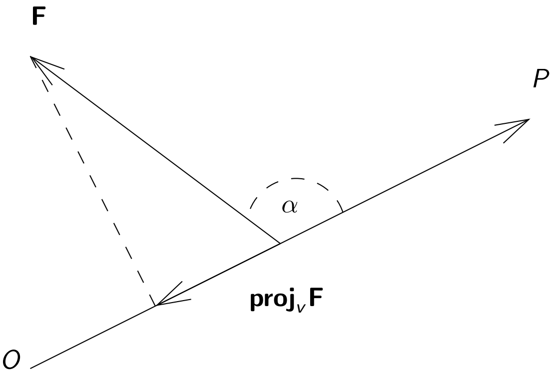
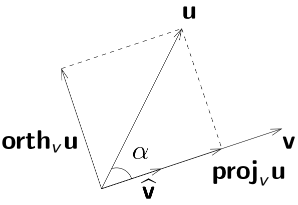


$$W = -|\mathbf{proj}_v\mathbf{F}|\,|\mathbf{OP}| = |\mathbf{F}||\mathbf{OP}|\cos\alpha\ .$$

Dot Product



- $\mathbf{u}$, $\mathbf{v}$ vectors, $\mathbf{v} \neq \mathbf{0}$.
- $\mathbf{proj}_v\mathbf{u}$: projection of $\mathbf{u}$ along $\mathbf{v}$;
- $\mathbf{orth}_v\mathbf{u}$: projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$;
- $\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$: unit vector along $\mathbf{v}$;
- $\mathbf{proj}_v\mathbf{u} = (comp_v\mathbf{u})\hat{\mathbf{v}}$
- $comp_v\mathbf{u}$: scalar projection of $\mathbf{u}$ onto $\mathbf{v}$;
- Dot product of u and v:

$$\mathbf{u} \cdot \mathbf{v} = (comp_v\mathbf{u})\,|\mathbf{v}|\ .$$

Properties of Dot Product

- If $\mathbf{v} = \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.
- If $\mathbf{u} \neq 0 \neq \mathbf{v}$, then $comp_v\mathbf{u} = |\mathbf{u}| \cos \alpha$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\, |\mathbf{v}|\, \cos \alpha$$

- If $\mathbf{u} \neq \mathbf{0} \neq \mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = 0 \Longleftrightarrow \mathbf{u} \perp \mathbf{v}\ .$$

- $\mathbf{u} \cdot \mathbf{v} = (\mathbf{proj}_v \mathbf{u}) \cdot \mathbf{v}$
- The dot product is linear in each argument:

$$(a\mathbf{u} + b\mathbf{w}) \cdot \mathbf{v} = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{w} \cdot \mathbf{v}$$

$$\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{u} \cdot \mathbf{w}$$

- Dot product is positive definite:

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0 \text{ and } \mathbf{v} \cdot \mathbf{v} = 0 \leftrightarrow \mathbf{v} = \mathbf{0}$$

Computations in Coordinates

- $Oxyz$: rectangular coordinate system
- $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$: unit vectors along fundamental directions.

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{j}=\mathbf{i}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$$

$$\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$$

- $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}=\langle u_1,u_2,u_3\rangle$, $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}=\langle v_1,v_2,v_3\rangle$,

$$\mathbf{u}\cdot\mathbf{v}=\langle u_1,u_2,u_3\rangle\cdot\langle v_1,v_2,v_3\rangle=u_1v_1+u_2v_2+u_3v_3\ .$$

- Example: $\langle 1,2,3\rangle\cdot\langle 6,5,4\rangle=1\cdot 6+2\cdot 5+3\cdot 4=28.$

Projections in Coordinates

$$\mathbf{u} = u_1 i + u_2 j + u_3 k = \langle u_1, u_2, u_3 \rangle, \; \mathbf{v} = v_1 i + v_2 j + v_3 k = \langle v_1, v_2, v_3 \rangle$$

$$comp_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$$

$$comp_{\langle 6,5,4 \rangle} \langle 1, 2, 3 \rangle = \frac{28}{\sqrt{77}} \; .$$

$$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|} \mathbf{v} = \frac{1}{\sqrt{77}} \langle 6, 5, 4 \rangle \; .$$

$$\mathbf{proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \widehat{\mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \; .$$

$$\mathbf{proj}_{\langle 6,5,4 \rangle} \langle 1, 2, 3 \rangle = \frac{28}{77} \langle 6, 5, 4 \rangle \; .$$

$$\mathbf{orth}_{\langle 6,5,4 \rangle} \langle 1, 2, 3 \rangle = \langle 1, 2, 3 \rangle - \mathbf{proj}_{\langle 6,5,4 \rangle} \langle 1, 2, 3 \rangle$$

Angles

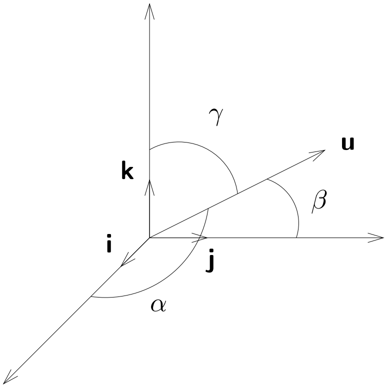
$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \rightarrow \alpha = \arccos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right)$$

Example:

Angle between $\langle 1, 2, 3\rangle$ and $\langle 6, 5, 4\rangle$:

$$\alpha = \arccos\left(\frac{28}{\sqrt{14}\,\sqrt{77}}\right) = \arccos\left(\frac{4}{\sqrt{22}}\right)$$

Direction Angles



$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \qquad \alpha = \angle(\mathbf{u}, \mathbf{i}) \qquad \beta = \angle(\mathbf{u}, \mathbf{j}) \qquad \gamma = \angle(\mathbf{u}, \mathbf{k}) \ .$$

$$\cos\alpha = \frac{\mathbf{u} \cdot \mathbf{i}}{|\mathbf{u}|\,|\mathbf{i}|} = \frac{u_1}{\sqrt{u_1^2 + u_2^2 + u_3^2}}$$

Similar for $\cos\beta$ and $\cos\gamma$. Then:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \ .$$