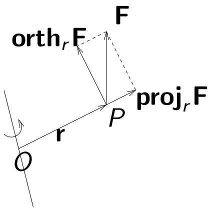
Cross Product

February 3, 2010

Rotational Effect

- Rigid rod OP, fixed at O, $\mathbf{r} = \mathbf{OP}$.
- Force $\mathbf{F}$ applied at P.



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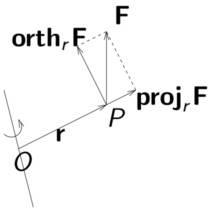

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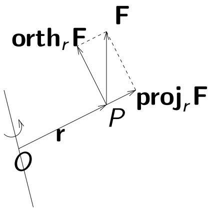

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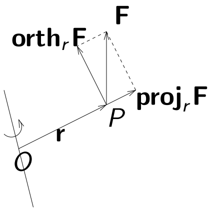

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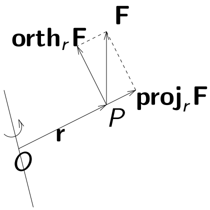

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  - Axis of rotation:

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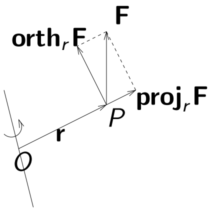

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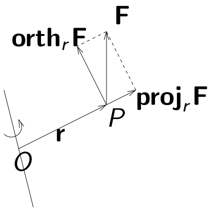

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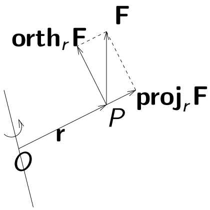

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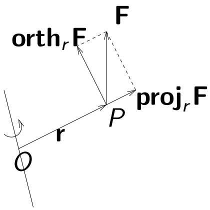

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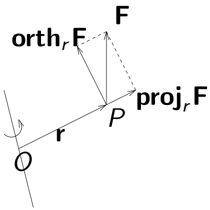

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Torque

Rotation in space $\Longleftrightarrow$ vector

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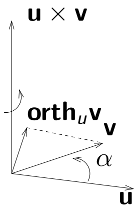
$$(\mathbf{r}, \mathbf{F}) \to \text{rotation} \to \text{vector} = \text{torque}, \boldsymbol{\tau}$$

- Support of direction: perpendicular to **r** and **F**;
- Direction: Right Hand Rule;
- Magnitude: $|\boldsymbol{\tau}| = |\mathbf{r}| \, |\mathbf{orth}_r \mathbf{F}|$

Cross Product

$$(\textbf{vector}, \textbf{vector}) \to \textbf{vector}$$



- If **u**, **v** are non-zero and non-collinear vectors:

$$\mathbf{u} \times \mathbf{v} \text{ is the vector determined by:}$$

  - Support of direction of $\mathbf{u} \times \mathbf{v}$: perpendicular to both **u** and **v**;
  - Direction of $\mathbf{u} \times \mathbf{v}$: given by Right Hand Rule;
  - Magnitude of $\mathbf{u} \times \mathbf{v}$: product of $|\mathbf{u}|$ and $|\mathbf{orth}_u \mathbf{v}|$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{orth}_u \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\alpha$$

- If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or **u** and **v** are collinear:

$$\mathbf{u} \times \mathbf{v} = \mathbf{0}\,.$$

Properties of Cross Product

$\mathbf{u}$, $\mathbf{v}$: non-zero vectors, $\alpha = \angle(\mathbf{u}, \mathbf{v})$:

$$|\mathbf{orth}_u \mathbf{v}| = |\mathbf{v}| \sin\alpha \Longrightarrow |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,\sin\alpha$$

Consequence: $|\mathbf{v} \times \mathbf{u}| = |\mathbf{u} \times \mathbf{v}|$

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- Cross product is anti-symmetric:

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- Cross product is linear in each argument:

$$\mathbf{u} \times (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \times \mathbf{v} + b\mathbf{u} \times \mathbf{w}$$

$$(a\mathbf{u} + b\mathbf{w}) \times \mathbf{v} = a\mathbf{u} \times \mathbf{v} + b\mathbf{w} \times \mathbf{v}$$

Cross Product in Coordinates

- $Oxyz$: rectangular coordinate system;
- $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$: unit vectors along fundamental directions.

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$$\mathbf{i} \times \mathbf{i} = \mathbf{0} \quad , \quad \mathbf{j} \times \mathbf{j} = \mathbf{0} \quad , \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad , \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad , \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad , \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad , \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Cross Product in Coordinates

- *Oxyz*: rectangular coordinate system;
- **i**, **j**, **k**: unit vectors along fundamental directions.

$$\mathbf{i}\times\mathbf{i}=\mathbf{0}\quad,\quad\mathbf{j}\times\mathbf{j}=\mathbf{0}\quad,\quad\mathbf{k}\times\mathbf{k}=\mathbf{0}$$

$$\mathbf{i}\times\mathbf{j}=\mathbf{k}\quad,\quad\mathbf{j}\times\mathbf{k}=\mathbf{i}\quad,\quad\mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k}\quad,\quad\mathbf{k}\times\mathbf{j}=-\mathbf{i}\quad,\quad\mathbf{i}\times\mathbf{k}=-\mathbf{j}$$

If $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}=\langle u_1,u_2,u_3\rangle$, $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}=\langle v_1,v_2,v_3\rangle$:

$$\mathbf{u}\times\mathbf{v}=(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\times(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})=$$

$$=(u_2v_3-u_3v_2)\mathbf{i}+(u_3v_1-u_1v_3)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}\ .$$

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$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad , \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad , \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = \langle v_1, v_2, v_3 \rangle$:

$$\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) =$$

$$= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \, .$$

$$\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example

$$\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

If $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 6, 5, 4 \rangle$, then

$$\mathbf{u} \times \mathbf{v} = \langle 1, 2, 3 \rangle \times \langle 6, 5, 4 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix} \mathbf{k} =$$

$$= (2 \cdot 4 - 3 \cdot 5)\mathbf{i} - (1 \cdot 4 - 3 \cdot 6)\mathbf{j} + (1 \cdot 5 - 2 \cdot 6)\mathbf{k} =$$

$$= -7\mathbf{i} + 14\mathbf{j} - 7\mathbf{k}\ .$$

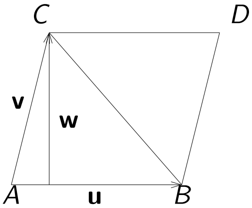
Applications of Cross Product

Vector perpendicular to given vectors $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \times \mathbf{v}$.
Example: Vector perpendicular to $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + \mathbf{k}$:

$$\begin{aligned} \mathbf{w} =& (\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \mathbf{i} \times \mathbf{j} + \mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{j} + \mathbf{j} \times \mathbf{k} = \\ =& \mathbf{k} - \mathbf{j} + \mathbf{0} + \mathbf{i} = \mathbf{i} - \mathbf{j} + \mathbf{k} \ . \end{aligned}$$

Applications of Cross Product

A, B, C points in space, $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$. Then



$$|\mathbf{w}| = |\mathbf{orth}_u\mathbf{v}| = \text{ distance from } C \text{ to } AB \text{ .}$$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{orth}_u\mathbf{v}|\,|\mathbf{u}| = 2\text{area}(ABC) = \text{area}(ABDC)$$

$|\mathbf{u} \times \mathbf{v}| =$ Area of parallelogram on sides $\mathbf{u}$ and $\mathbf{v}$.
Example: Area of triangle $A(1,2,3)$, $B(2,3,1)$, $C(3,1,2)$

$$\text{Area}(ABC) = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2}|\langle 1,1,-2\rangle \times \langle 2,-1,-1\rangle| =$$

$$= \frac{1}{2}|\langle -3,-3,-3\rangle| = \frac{3\sqrt{3}}{2} \text{ .}$$

Scalar Triple Product

- A, B, C, D points in space;
- $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$, $\mathbf{w} = \mathbf{AD}$;
- $R = R(\mathbf{u}, \mathbf{v}, \mathbf{w})$: box on sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

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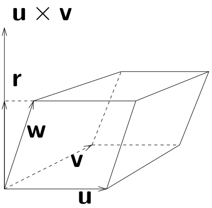

$$\text{Vol}(R) = |\mathbf{u} \times \mathbf{v}||\mathbf{r}| = |\mathbf{u} \times \mathbf{v}|\,|\mathbf{proj}_{u \times v}\mathbf{w}| = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|\ .$$

- $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$: scalar triple product.
- If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Orientations of Space

- The following are equivalent:
  - Every vector in space can be decomposed along **u**, **v**, **w**;
  - The box $R(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is non-degenerate;
  - $Vol(R(\mathbf{u}, \mathbf{v}, \mathbf{w})) \neq 0$;
  - $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \neq 0$.

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  - $Vol(R(\mathbf{u}, \mathbf{v}, \mathbf{w})) \neq 0$;
  - $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \neq 0$.
- If any of the above is valid: $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a frame.
- Rectangular coordinate system $\rightarrow$ fundamental frame $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$
- Consistent Right Hand Rule ($\mathbf{w} = \mathbf{u} \times \mathbf{v}$) if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0$$

The frame $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is positively oriented if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0$