Lines and Planes

February 8, 2010

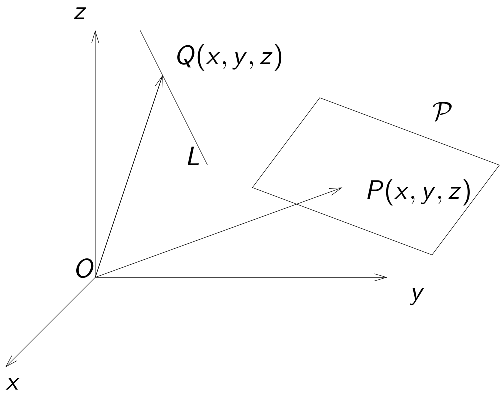
Main Questions

What condition or conditions should

- the position vector
- the coordinates

of a point satisfy
for the point to be
on a specific

- line L
- plane $\mathcal{P}$?



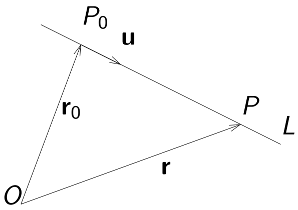
Condition(s) in terms of:

- position vector $\Rightarrow$ vectorial equations;
- coordinates $\Rightarrow$ scalar equations.

Line from Point and Direction

- Point P_0, with position vector $\mathbf{r}_0$;
- Direction $\leftrightarrow$ non-zero vector $\mathbf{u}$.

L: line with direction $\mathbf{u}$, passing through P_0



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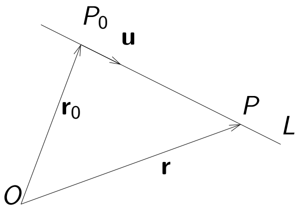

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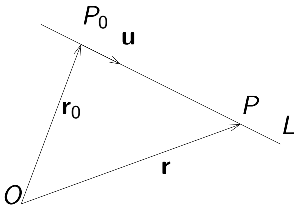

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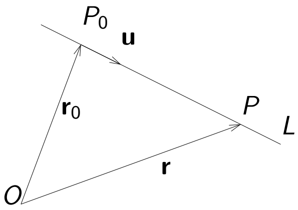

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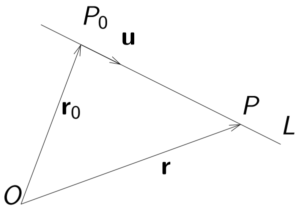
$\mathbf{r} - \mathbf{r}_0 = t\mathbf{u}$ for some real number t



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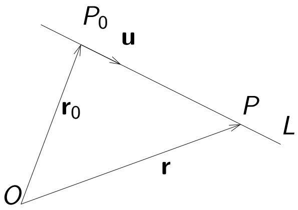

Parametric vectorial equation: $\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}}$ for some real number t

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- Point $P_0(x_0, y_0, z_0)$,
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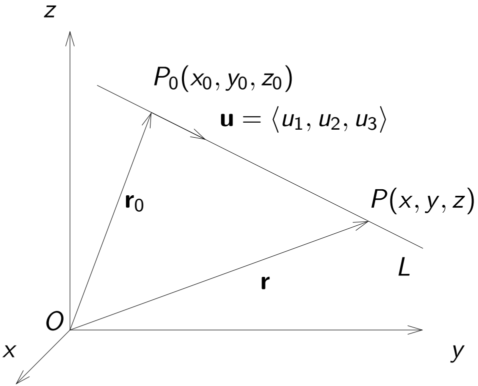

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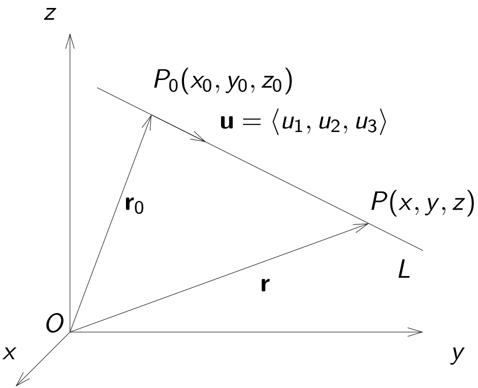

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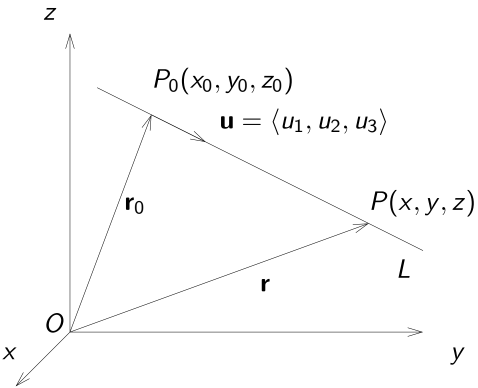

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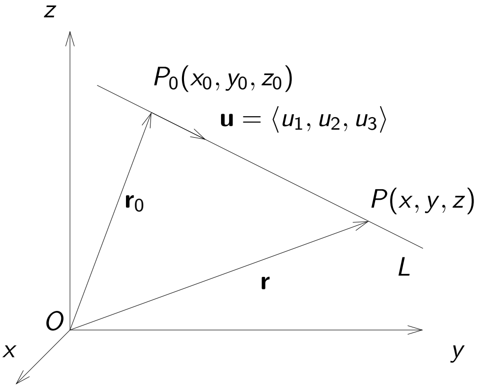

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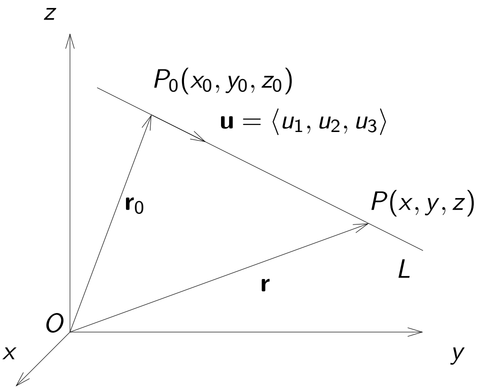
Parametric scalar equations:

$$\begin{cases} x = x_0 + tu_1 \\ y = y_0 + tu_2 \\ z = z_0 + tu_3 \end{cases}$$

for some real parameter t



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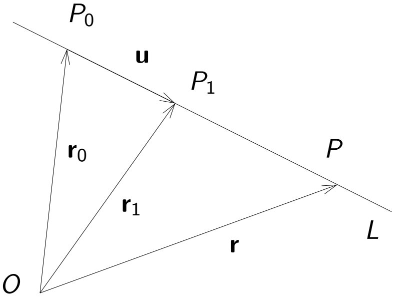

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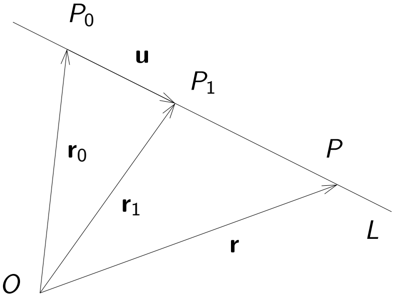

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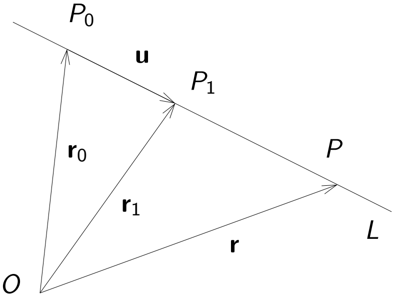
Equivalent equation:
$\mathbf{r} = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$



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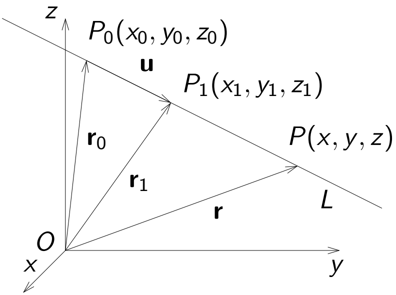

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Plane from Point and Normal

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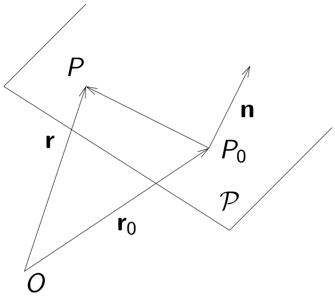

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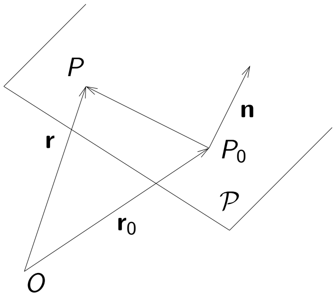

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Implicit vectorial equation:

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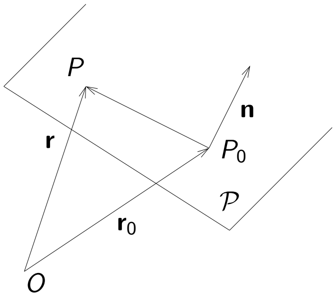

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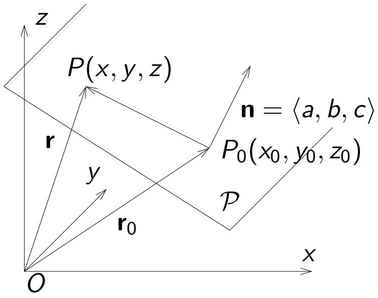

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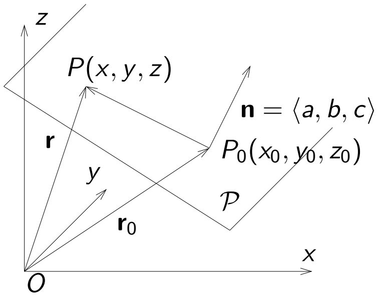
$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$



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General equation of a plane:

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Coefficients a, b, c: components of normal to the plane,

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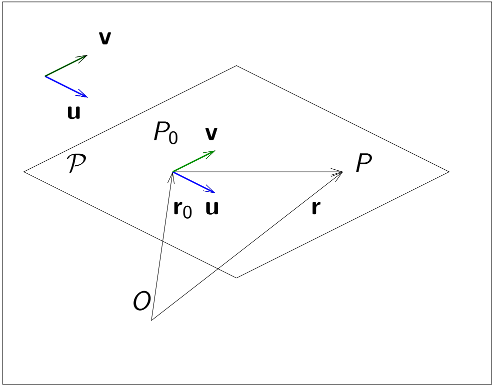
Plane from Point and two Directions

Point P_0, with position vector $\mathbf{r}_0$;
Non-parallel directions $\mathbf{u}$ and $\mathbf{v}$.

$\mathcal{P}$: plane through P_0 and parallel to both $\mathbf{u}$ and $\mathbf{v}$.



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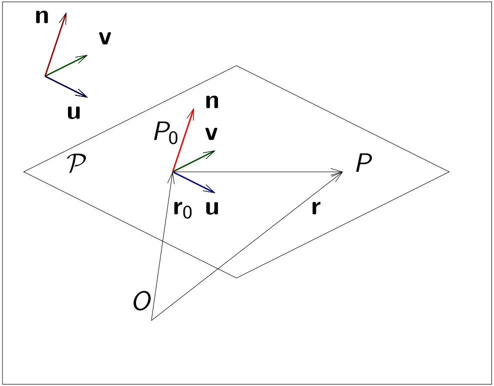
Normal direction $\mathbf{n} = \mathbf{u} \times \mathbf{v} \neq \mathbf{0}$

Implicit vectorial equation:

$$P(\mathbf{r}) \text{ is on } \mathcal{P} \iff \boxed{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0}$$

Interpretation:

$$\mathrm{Vol}(R(\mathbf{r} - \mathbf{r}_0, \mathbf{u}, \mathbf{v})) = 0$$



Plane from Point and two Directions

Point P_0, with position vector $\mathbf{r}_0$;
Non-parallel directions $\mathbf{u}$ and $\mathbf{v}$.

$\mathcal{P}$: plane through P_0 and parallel to both $\mathbf{u}$ and $\mathbf{v}$.

$P(\mathbf{r})$ is on the plane $\mathcal{P}$ $\leftrightarrow$

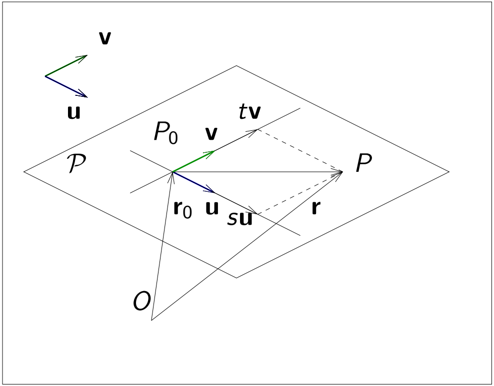
$\mathbf{P_0P}$ is a combination of $\mathbf{u}$, $\mathbf{v}$ $\leftrightarrow$

There are scalars s, t such that
$\mathbf{r} - \mathbf{r}_0 = s\mathbf{u} + t\mathbf{v}$ $\leftrightarrow$

Parametric vectorial equation:

$$\boxed{\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}}$$

for some parameters s and t



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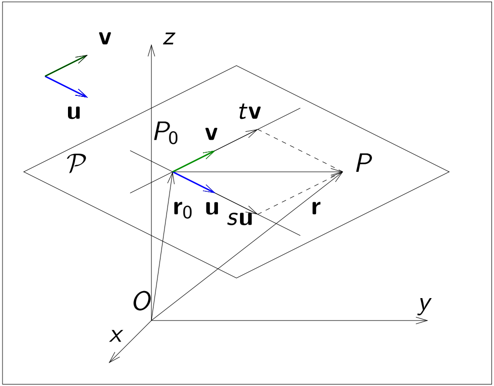
$P_0(x_0, y_0, z_0)$, $P(x, y, z)$
$\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Parametric scalar equations:

$$\begin{cases} x = & x_0 + su_1 + tv_1 \\ y = & y_0 + su_2 + tv_2 \\ z = & z_0 + su_3 + tv_3 \end{cases}$$

for s, t real parameters.

$\mathcal{P}$: plane through P_0 and parallel to both $\mathbf{u}$ and $\mathbf{v}$.



Example

$P_0(1,2,3)$, $\mathbf{u} = \langle -1, 0, 2 \rangle$, $\mathbf{v} = \langle 0, -2, 1 \rangle$.

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$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$4(x-1) + 1(y-2) + 2(z-3) = 0 \iff 4x + y + 2z = 12$$

Example

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Parametric vectorial equation:

$$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + s\langle -1, 0, 2 \rangle + t\langle 0, -2, 1 \rangle$$

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$$4x + y + 2z = 12\ .$$

Parametric vectorial equation:

$$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + s\langle -1, 0, 2 \rangle + t\langle 0, -2, 1 \rangle$$

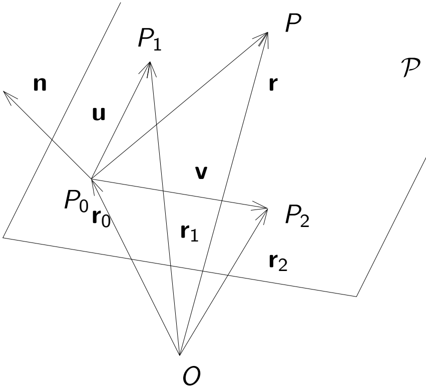
Parametric scalar equations:

$$\begin{cases} x &= 1 - s \\ y &= 2 - 2t \\ z &= 3 + 2s + t \end{cases} \qquad s, t \text{ real parameters.}$$

Plane from Three Points

Three non-collinear points
$P_0(\mathbf{r}_0)$, $P_1(\mathbf{r}_1)$, $P_2(\mathbf{r}_2)$

Plane $\mathcal{P}$
passing through P_0, P_1, and P_2.



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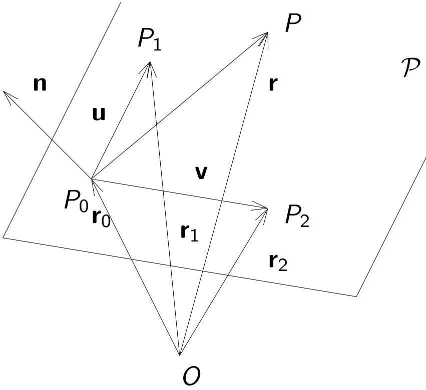
Passing through $P_0(\mathbf{r}_0)$
Parallel to
$\mathbf{u} = \mathbf{P}_0\mathbf{P}_1 = \mathbf{r}_1 - \mathbf{r}_0$
$\mathbf{v} = \mathbf{P}_0\mathbf{P}_2 = \mathbf{r}_2 - \mathbf{r}_0$.
Normal $\mathbf{n} = \mathbf{u} \times \mathbf{v} = (\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)$
Implicit vectorial equation:

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$

$$\boxed{(\mathbf{r} - \mathbf{r}_0) \cdot [(\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)] = 0}$$

$$\text{Vol}(R(\mathbf{P}_0\mathbf{P}, \mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2)) = 0$$



Plane from Three Points

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$P_0(\mathbf{r}_0)$, $P_1(\mathbf{r}_1)$, $P_2(\mathbf{r}_2)$

Plane $\mathcal{P}$
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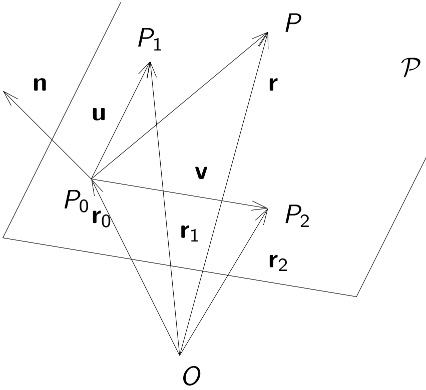
Implicit vectorial equation:

$$(\mathbf{r} - \mathbf{r}_0) \cdot [(\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)] = 0$$

$P_0(x_0, y_0, z_0)$, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$:
$P(x, y, z)$ is on plane $\mathcal{P}$:

Implicit scalar equation:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\ x_2 - x_0 & y_2 - y_0 & z_2 - z_0 \end{vmatrix} = 0 \ .$$



Example

$P_0(a,0,0)$, $P_1(0,b,0)$, $P_2(0,0,c)$ distinct points on axes.
$\mathcal{P}$: plane through P_0, P_1, P_2 = plane with given intercepts.

$\mathcal{P}$: parallel to
$\mathbf{P}_0\mathbf{P}_1 = \langle -a, b, 0\rangle$, $\mathbf{P}_0\mathbf{P}_2 = \langle -a, 0c\rangle$
Normal to: $\mathbf{n} = \mathbf{P}_0\mathbf{P}_1 \times \mathbf{P}_0\mathbf{P}_2$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}\ .$$

Implicit scalar equation of plane:

$$\langle x-a, y, z\rangle \cdot \langle bc, ac, ab\rangle = 0$$

$$bcx + acy + abz = abc \iff \boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$$