Geometry of Lines and Planes

February 10, 2010

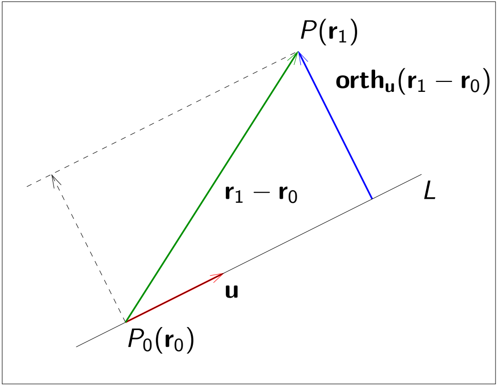
Main Questions

- Geometric objects:
  - Points: $P(\mathbf{r})$.
  - Lines: L: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$
  - Planes: $\mathcal{P}$: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$
- Relationships/Geometric Quantities:
  - Parallelism
  - Perpendicularity
  - Angles
  - Distances
  - Intersections

Point and line

Point $P(\mathbf{r}_1)$ Line L : $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$

Distance from P to L:



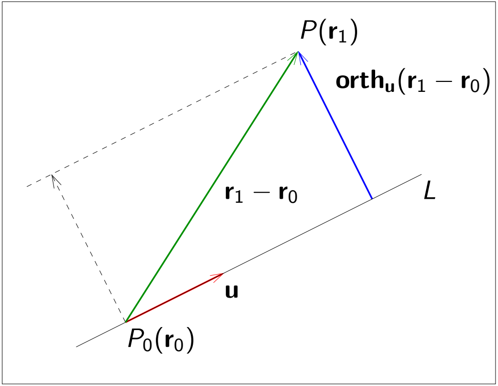
Point and line

Point $P(\mathbf{r}_1)$ Line $L: \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$

Distance from P to L:

$$d(P, L) = |\mathbf{orth}_{\mathbf{u}}(\mathbf{r}_1 - \mathbf{r}_0)|$$

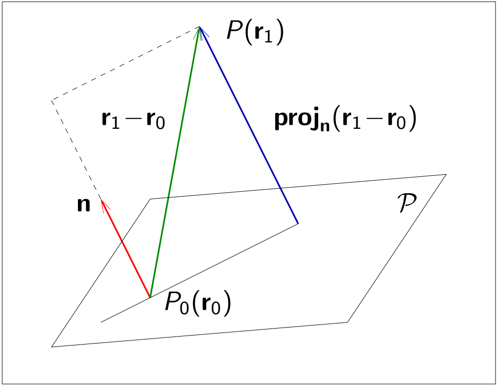
$$\boxed{d(P, L) = \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{u}|}{|\mathbf{u}|}}$$



Point and plane

Point $P(\mathbf{r}_1)$ Plane $\mathcal{P}$: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$

Distance from P to $\mathcal{P}$:



Point and plane

Point $P(\mathbf{r}_1)$ Plane $\mathcal{P}$: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$

Distance from P to $\mathcal{P}$:

$$d(P, \mathcal{P}) = |\mathbf{proj}_{\mathbf{n}}(\mathbf{r}_1 - \mathbf{r}_0)|$$

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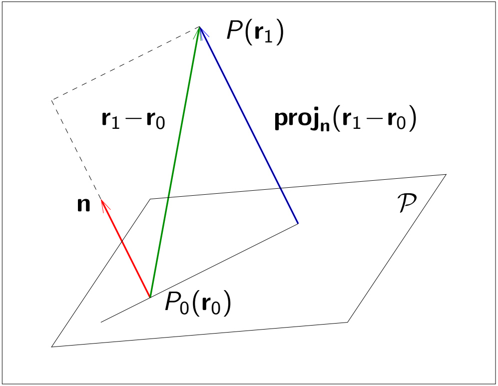

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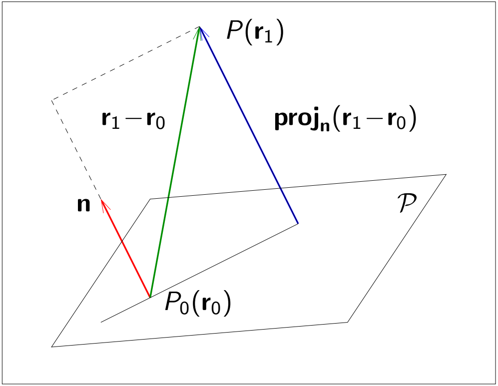
Scalar equation: $P(x_1, y_1, z_1)$

$$\mathcal{P}: ax + by + cz + d = 0$$



Point and plane

Point $P(\mathbf{r}_1)$ Plane $\mathcal{P}$: $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n}=0$

Distance from P to $\mathcal{P}$:

$$d(P,\mathcal{P}) = |\mathbf{proj}_{\mathbf{n}}(\mathbf{r}_1-\mathbf{r}_0)|$$

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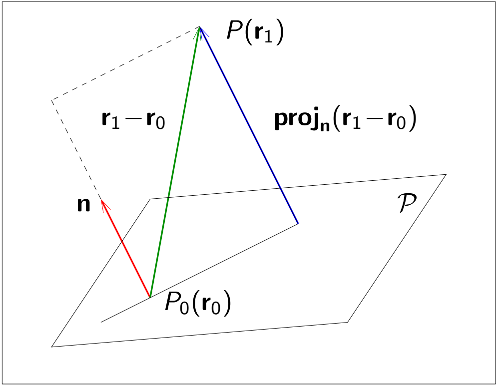
Scalar equation: $P(x_1, y_1, z_1)$

$$\mathcal{P} : ax+by+cz+d=0$$

$$\mathbf{n} = \langle a,b,c\rangle$$

$$\boxed{\text{Distance} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

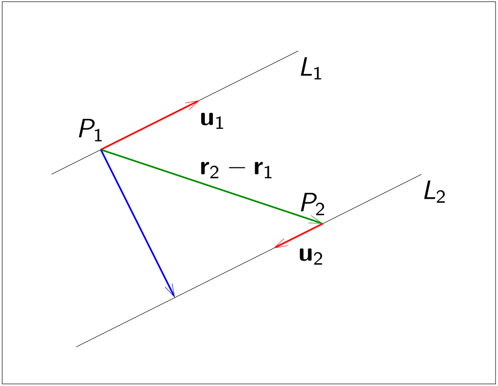


Parallel lines

Lines

$$L_1 : \quad \mathbf{r} = \mathbf{r}_1 + t\mathbf{u}_1 \qquad L_2 : \quad \mathbf{r} = \mathbf{r}_2 + s\mathbf{u}_2$$

Parallel lines



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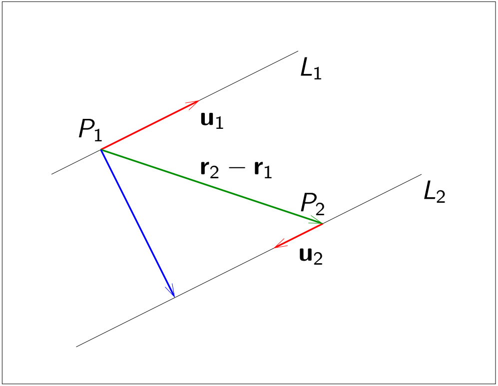
Parallel lines

$L_1 || L_2 \iff \mathbf{u}_1,\ \mathbf{u}_2$ collinear $\iff$

$$\boxed{\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{0}}$$



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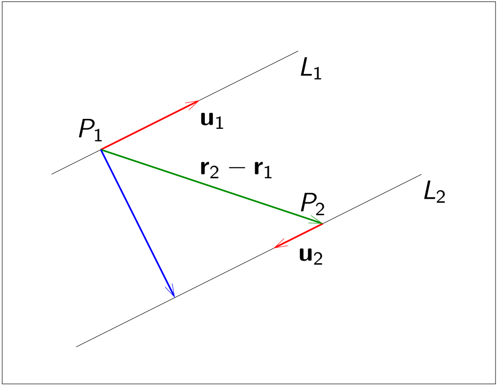

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$$d = d(L_1, L_2) = |\mathbf{orth}_{\mathbf{u}_1}(\mathbf{r}_2 - \mathbf{r}_1)|$$

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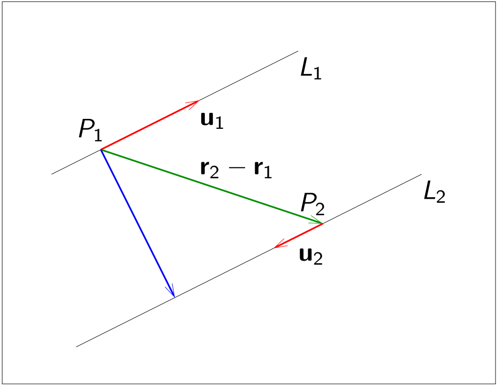

Angle between lines

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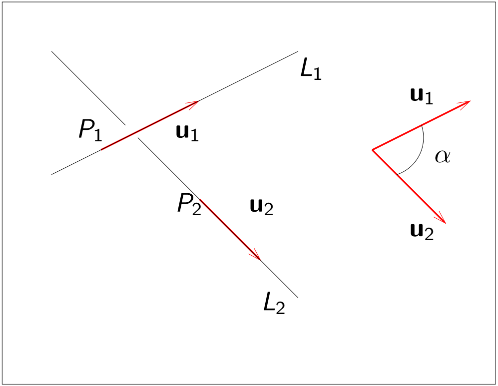
Perpendicular lines



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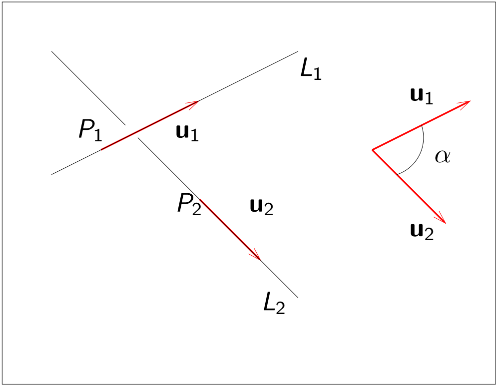

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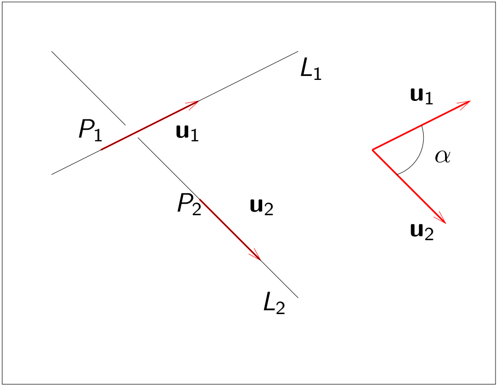

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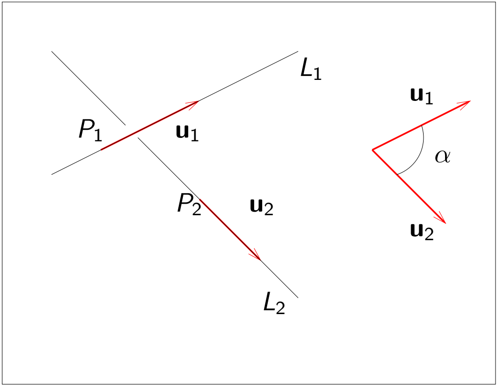
Angle between lines

α: angle between L_1, $L_2 \iff$

α: acute angle $\mathbf{u}_1$, $\mathbf{u}_2$



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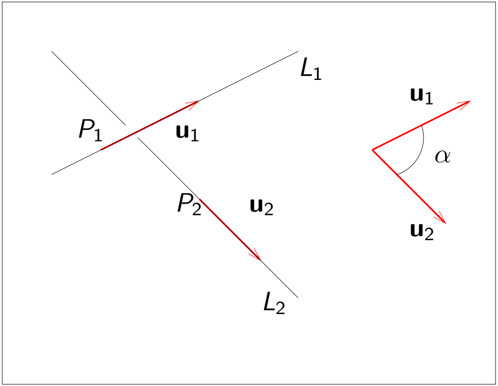

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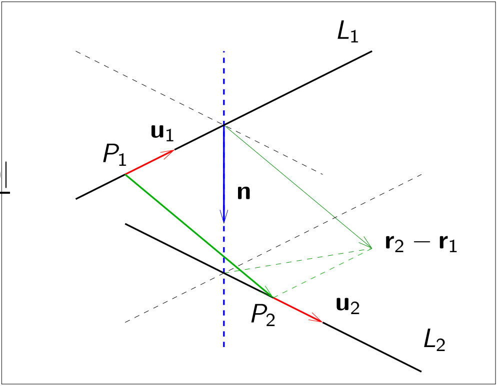
Skew lines

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 \neq \mathbf{0}$$

Distance:

$$d(L_1, L_2) = |\mathbf{proj}_{\mathbf{n}}(\mathbf{r}_2 - \mathbf{r}_1)| =$$

$$= \boxed{\frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{n}|}{|\mathbf{n}|}} = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{u}_1 \times \mathbf{u}_2)|}{|\mathbf{u}_1 \times \mathbf{u}_2|}$$



Intersecting lines: $d(L_1, L_2) = 0$

$$\mathbf{u}_1 \times \mathbf{u}_2 \neq \mathbf{0}$$

$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{u}_1 \times \mathbf{u}_2) = 0$$

Distance between lines

Lines

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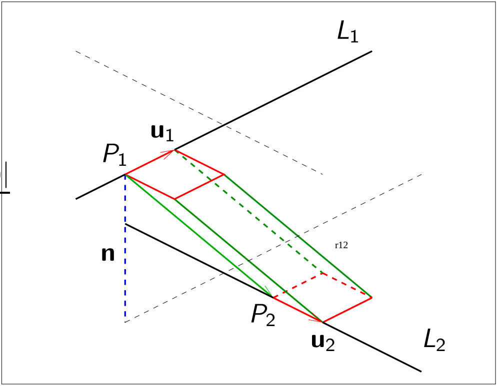

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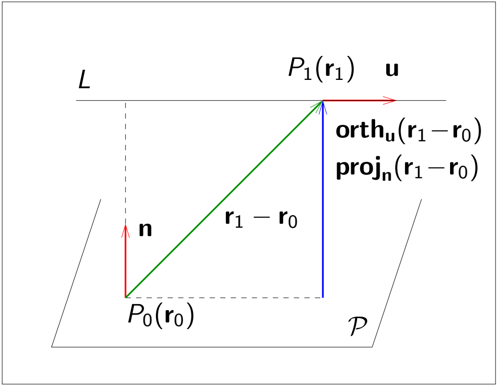
$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{u}_1 \times \mathbf{u}_2) = 0$$

Parallel line and plane

Line L : $\mathbf{r} = \mathbf{r}_1 + t\mathbf{u}$ Plane $\mathcal{P}$: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$

Line parallel to plane



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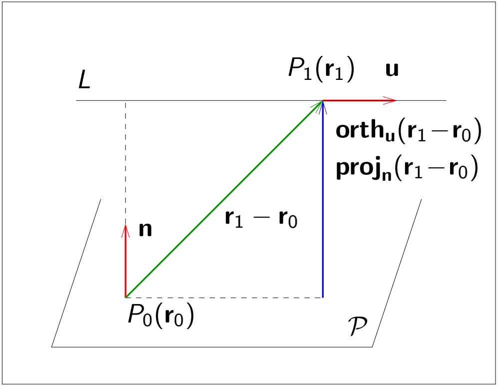

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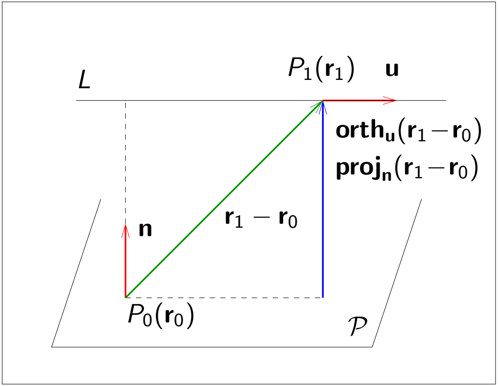
Distance from L to $\mathcal{P}$:

$$d(L, \mathcal{P}) = d(P_1, \mathcal{P})$$



Parallel line and plane

Line L : $\mathbf{r} = \mathbf{r}_1 + t\mathbf{u}$

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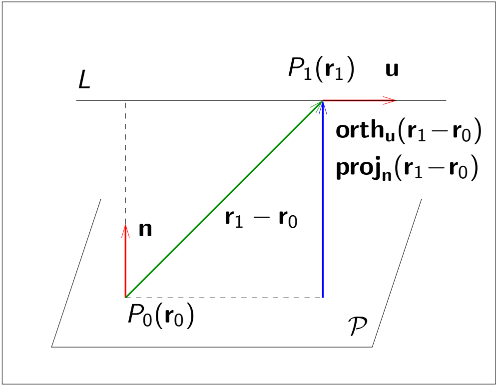
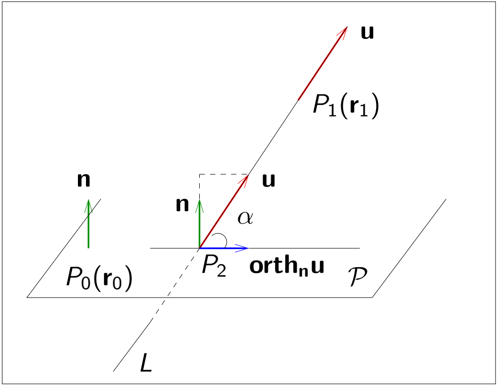
$$d = \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{u}|}{|\mathbf{u}|} = \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}|}{|\mathbf{n}|}$$



Angle between line and plane

Line L : $\mathbf{r} = \mathbf{r}_1 + t\mathbf{u}$ Plane $\mathcal{P}$: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$

Line perpendicular to plane



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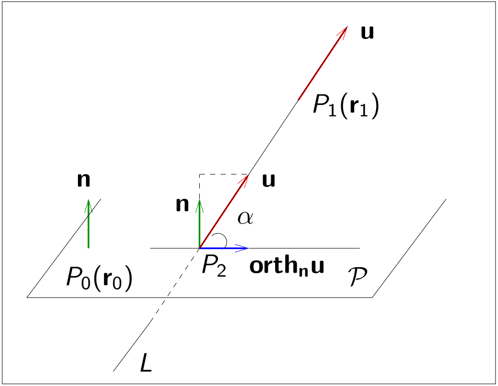

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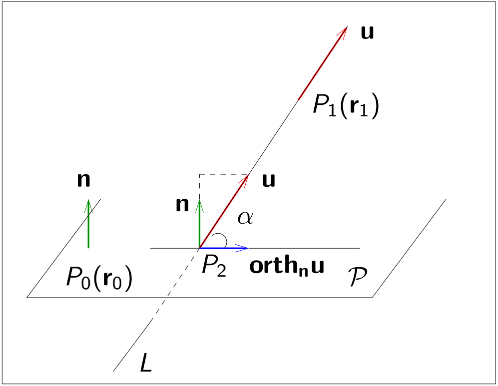

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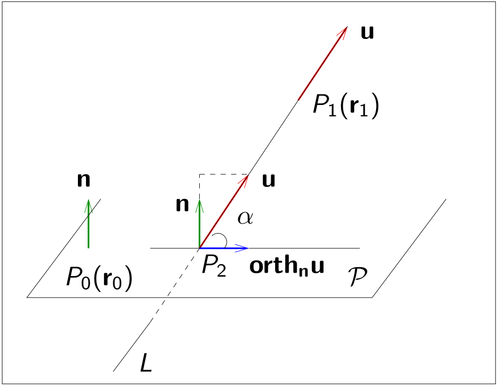

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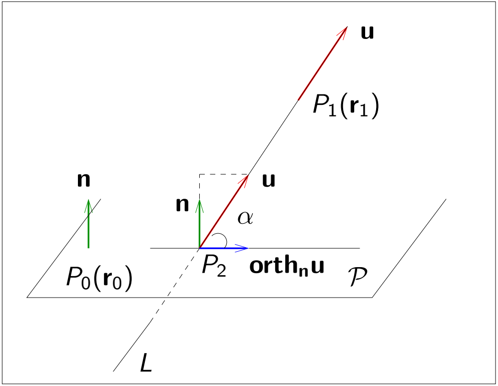
Intersection:



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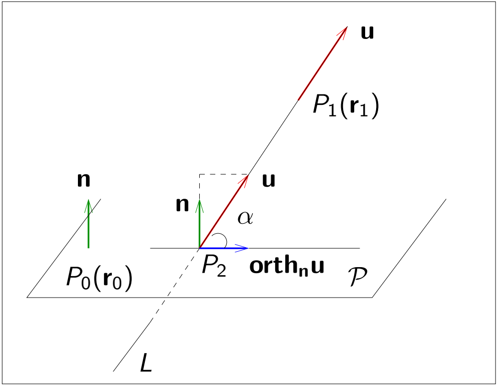


Intersection:

$$(\mathbf{r}_1 + t\mathbf{u} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$

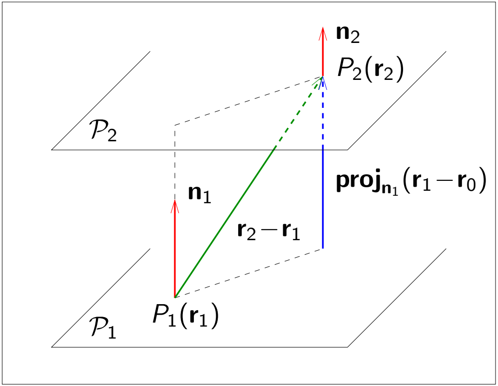
$$(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n} + t\mathbf{u} \cdot \mathbf{n} = 0$$

Solve for t.

Parallel planes

Planes $\mathcal{P}_1 : \quad (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 = 0$ $\qquad \mathcal{P}_2 : \quad (\mathbf{r} - \mathbf{r}_2) \cdot \mathbf{n}_2 = 0$

Parallel planes:



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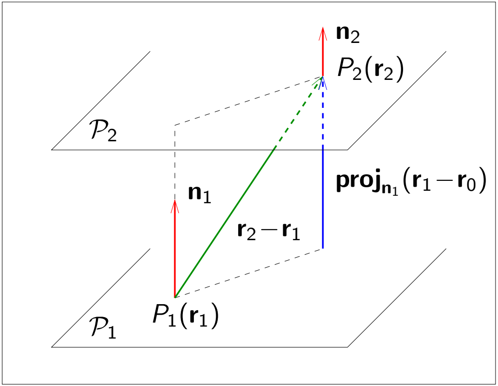

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Distance between parallel planes:

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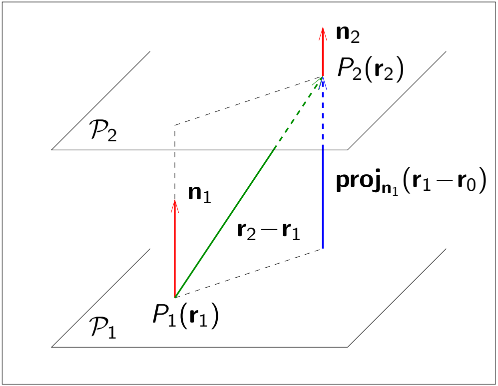


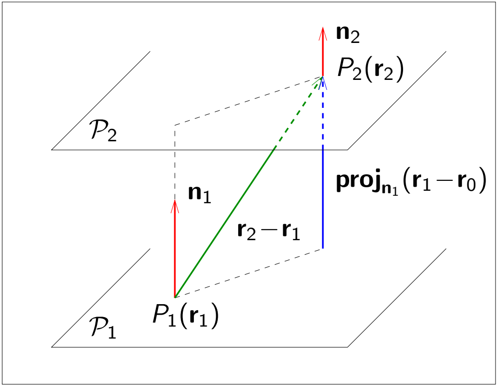
Parallel planes

Planes $\mathcal{P}_1: \quad (\mathbf{r}-\mathbf{r}_1)\cdot\mathbf{n}_1 = 0 \qquad \mathcal{P}_2: \quad (\mathbf{r}-\mathbf{r}_2)\cdot\mathbf{n}_2 = 0$

Parallel planes:

$\mathcal{P}_1 || \mathcal{P}_2 \iff \mathbf{n}_1,\ \mathbf{n}_2$ collinear $\iff$

$$\boxed{\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}}$$



Distance between parallel planes:

$$d(\mathcal{P}_1,\mathcal{P}_2) = |\mathbf{proj}_{\mathbf{n}_1}(\mathbf{r}_2 - \mathbf{r}_1)| =$$

$$= \boxed{\frac{|(\mathbf{r}_2 - \mathbf{r}_1)\cdot\mathbf{n}_1|}{|\mathbf{n}_1|}}$$

Scalar equations:

$\mathcal{P}_1: \quad ax + by + cz = d_1$

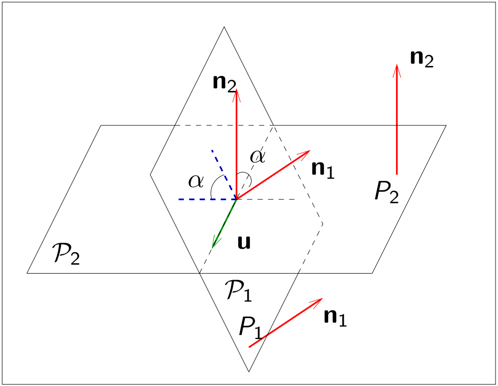
$\mathcal{P}_2: \quad ax + by + cz = d_2$

$$\boxed{d(\mathcal{P}_1,\mathcal{P}_2) = \frac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}}}$$

Angle between planes

Planes $\mathcal{P}_1 : \quad (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 = 0 \qquad \mathcal{P}_2 : \quad (\mathbf{r} - \mathbf{r}_2) \cdot \mathbf{n}_2 = 0$

Angle α between planes:



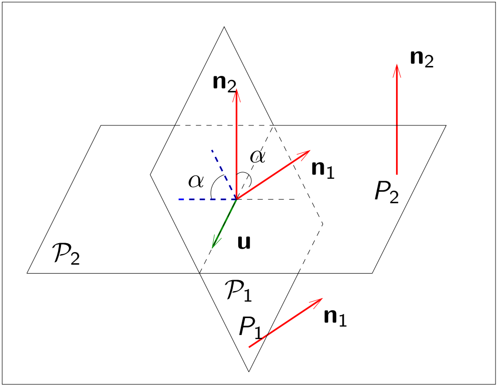
Angle between planes

Planes $\mathcal{P}_1 : \quad (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 = 0 \qquad \mathcal{P}_2 : \quad (\mathbf{r} - \mathbf{r}_2) \cdot \mathbf{n}_2 = 0$

Angle α between planes:

$$\alpha = \text{acute angle}(\mathbf{n}_1, \mathbf{n}_2)$$

$$\alpha = \arccos\left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$



Angle between planes

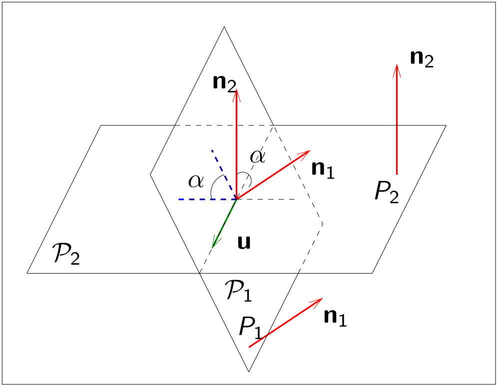
Planes $\mathcal{P}_1$: $(\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 = 0$ $\mathcal{P}_2$: $(\mathbf{r} - \mathbf{r}_2) \cdot \mathbf{n}_2 = 0$

Angle α between planes:

$$\alpha = \text{acute angle}(\mathbf{n}_1, \mathbf{n}_2)$$

$$\alpha = \arccos\left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$

Perpendicular planes:



Angle between planes

Planes $\mathcal{P}_1 : \quad (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 = 0$ $\qquad \mathcal{P}_2 : \quad (\mathbf{r} - \mathbf{r}_2) \cdot \mathbf{n}_2 = 0$

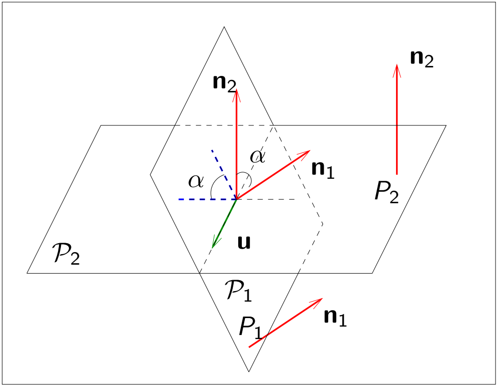
Angle α between planes:

$$\alpha = \text{acute angle}(\mathbf{n}_1, \mathbf{n}_2)$$

$$\alpha = \arccos\left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$

Perpendicular planes:

$$\alpha = \frac{\pi}{2} \iff \boxed{\mathbf{n}_1 \cdot \mathbf{n}_2 = 0}$$



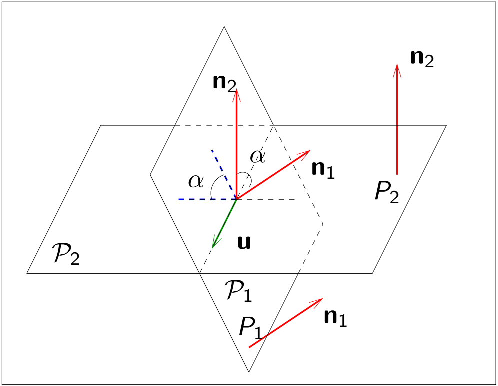
Angle between planes

Planes $\mathcal{P}_1: \quad (\mathbf{r}-\mathbf{r}_1)\cdot\mathbf{n}_1 = 0$ $\qquad \mathcal{P}_2: \quad (\mathbf{r}-\mathbf{r}_2)\cdot\mathbf{n}_2 = 0$

Angle α between planes:

$$\alpha = \text{acute angle}(\mathbf{n}_1, \mathbf{n}_2)$$

$$\alpha = \arccos\left(\frac{|\mathbf{n}_1\cdot\mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$



Perpendicular planes:

$$\alpha = \frac{\pi}{2} \iff \boxed{\mathbf{n}_1\cdot\mathbf{n}_2 = 0}$$

Direction of line of intersection:

Angle between planes

Planes $\mathcal{P}_1: \quad (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 = 0$ $\qquad \mathcal{P}_2: \quad (\mathbf{r} - \mathbf{r}_2) \cdot \mathbf{n}_2 = 0$

Angle α between planes:

$$\alpha = \text{acute angle}(\mathbf{n}_1, \mathbf{n}_2)$$

$$\alpha = \arccos\left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$

Perpendicular planes:

$$\alpha = \frac{\pi}{2} \Longleftrightarrow \boxed{\mathbf{n}_1 \cdot \mathbf{n}_2 = 0}$$

Direction of line of intersection:

$$\mathbf{u} = \mathbf{n}_1 \times \mathbf{n}_2$$