Parametric Curves

February 10, 2010

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- Functions $[a, b] \to \mathbb{R}^2$, $[a, b] \to \mathbb{R}^3$
- Scalar input (parameter)
- Vectorial output (position vector)
- t: parameter
- $\mathbf{r}$: parametrization

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- $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leqslant t \leqslant b$
- $x, y, z\colon [a, b] \to \mathbb{R}$, coordinate functions

Example

- $\mathbf{r} \colon \mathbb{R} \to \mathbb{R}^3$,
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$$
- Line through $P(1, 4, 3)$
direction $\mathbf{u} = \langle -1, 2, 0 \rangle$:

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- Parametrization of a curve: not unique!
- $t = 1 - s = \varphi(s)$: reparametrization
- $\mathbf{p}(s) = \mathbf{r}(\varphi(s))$
- $\mathbf{p}(s) = \langle s, 6 - 2s, 3 \rangle$

Describe the curve:

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Tornado!

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Close in Euclidean distance $\iff$ close in postman distance

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \quad , \quad \mathbf{u} = \langle u_1, u_2, u_3 \rangle$$

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{u} \iff \begin{cases} \lim_{t \to a} x(t) = u_1 \\ \lim_{t \to a} y(t) = u_2 \\ \lim_{t \to a} z(t) = u_3 \end{cases}$$

Continuity

$\mathbf{r}\colon [a,b] \to \mathbb{R}^3$ is continuous at t_0 if

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Component-wise:
$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous at $t_0 \iff$
$x, y, z \colon [a, b] \to \mathbb{R}$ are ALL continuous at t_0.

Derivatives

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^3 \quad , \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Average rate of change:

$$\frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0} = \frac{\text{Displacement}}{\text{Time}} = \text{Average Velocity}$$

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$\mathbf{r}(t)$ vector $\implies$ $\mathbf{r}'(t)$ vector
Derivatives: $\mathbf{r}'(t)$, $\mathbf{r}''(t) = (\mathbf{r}'(t))'$ (acceleration), ...

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

Example

Coordinate curves for spherical coordinates:
Fix $\rho = \rho_0$ and $\phi = \phi_0$.

$$\mathbf{r}(t) = \langle \rho_0 \sin\phi_0 \cos t, \rho_0 \sin\phi_0 \sin t, \rho_0 \cos\phi_0 \rangle$$

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Then

$$\mathbf{r}'(t) = \langle -\rho_0 \sin\phi_0 \sin t, \rho_0 \sin\phi_0 \cos t, 0 \rangle$$

Define

$$\widehat{\mathbf{e}}_\theta(\rho_0, \phi_0, \theta_0) = \frac{1}{|\mathbf{r}'(\theta_0)|}\,\mathbf{r}'(\theta_0) = \langle -\sin\theta_0, \cos\theta_0, 0 \rangle$$

Similar constructions for $\widehat{\mathbf{e}}_\rho(\rho_0, \phi_0, \theta_0)$ and $\widehat{\mathbf{e}}_\phi(\rho_0, \phi_0, \theta_0)$

Compute $\qquad (\widehat{\mathbf{e}}_\rho \times \widehat{\mathbf{e}}_\theta) \cdot \widehat{\mathbf{e}}_\phi$

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Product Rules:

$$[f(t)\mathbf{r}(t)]' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$$
$$[\mathbf{u}(t) \cdot \mathbf{v}(t)]' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$
$$[\mathbf{u}(t) \times \mathbf{v}(t)]' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

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Example:

$$\frac{d|\mathbf{r}(t)|}{dt} = [\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}]' = [\sqrt{\square}]' = \frac{1}{2\sqrt{\square}}\square' = \frac{1}{2\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}}[\mathbf{r}(t)\cdot\mathbf{r}(t)]' =$$

$$= \frac{1}{2|\mathbf{r}(t)|}[\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t)] = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}$$

Application

$$|\mathbf{r}|' = \frac{\mathbf{r} \cdot \mathbf{r}'}{|\mathbf{r}|}$$

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Velocity $\perp$ Position $\Longleftrightarrow$ Velocity vector $\mathbf{r}'(t)$ tangent to sphere.

How about the acceleration?

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$$\mathbf{r}\cdot\mathbf{r}' \equiv 0 \Longrightarrow [\mathbf{r}\cdot\mathbf{r}']' \equiv 0 \iff \mathbf{r}'\cdot\mathbf{r}' + \mathbf{r}\cdot\mathbf{r}'' \equiv 0 \Longrightarrow \mathbf{r}\cdot\mathbf{r}'' = -|\mathbf{r}'|^2 \leqslant 0$$

Acceleration vector $\mathbf{r}''$ points inside the sphere.

Definite Integrals

$\mathbf{r}\colon [a,b] \to \mathbb{R}^3$

- Division $a = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots < t_n = b$
- Sample points $t_k \leqslant s_k \leqslant t_{k+1}$
- Riemann sum:

$$\sum_{k=0}^{n-1} (t_{k+1} - t_k)\mathbf{r}(s_k)$$

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- Result:

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- Result: a vector.

Properties

- Component-wise: if $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\int_a^b \mathbf{r}(t)\ dt = \langle \int_a^b x(t)\ dt, \int_a^b y(t)\ dt, \int_a^b z(t)\ dt \rangle$$

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- Derivative $\Longrightarrow$ Total change

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{r}'(\tau)\ d\tau$$

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$$\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t (\mathbf{v}_0 - g\tau\mathbf{k})\, d\tau = \mathbf{r}_0 + \left(\int_0^t d\tau\right)\mathbf{v}_0 - \left(\int_0^t g\tau\ d\tau\right)\mathbf{k} \Longrightarrow$$

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}_0 - \frac{1}{2}gt^2\mathbf{k}}$$

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Parabola in the plane determined by $\mathbf{v}_0$ and $\mathbf{k}$.