Limits and Continuity

February 26, 2010

Limits

$f: D \to \mathbb{R}$, with D a subset of the plane.
P_0: point in plane such that:

- f is defined arbitrarily close to P_0;
- f is not necessarily defined at P_0.

Example:

$$f: D = \mathbb{R}^2 \setminus \{P_0(0,0)\} \to \mathbb{R}, \qquad f(x,y) = \frac{x^2 y}{x^2 + y^2}$$

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Question: What happens to $f(Q)$ as Q gets closer to P_0?

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$$f(Q) \longrightarrow ? \text{ as } Q \to P_0(0,0)$$

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Numerical approach:

$$Q_1(0.1, 0.1) \Longrightarrow f(Q_1) = f(0.1, 0.1) \simeq 0.05$$
$$Q_2(0.01, -0.02) \Longrightarrow f(Q_2) = f(0.01, -0.02) \simeq -0.004$$
$$Q_3(-0.003, 0.001) \Longrightarrow f(Q_3) = f(-0.003, 0.001) \simeq 0.0009$$

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Numerical data suggests:

$$f(Q) \text{ is closer and closer to } 0 \text{ as } Q \to P_0(0, 0)$$

Limit of a function at a point

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Definition

A value L (finite or infinite) is the limit of f at P_0 if we can keep the values of $f(Q)$ as close to L as we want by keeping Q close enough to P_0, but not equal to P_0.

Notation:

$$L = \lim_{Q \to P_0} f(Q) \quad \text{or} \quad L = \lim_{(x,y) \to (x_0,y_0)} f(x, y)$$

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Remarks:

- $L = \infty$: close to $\infty \iff$ large;
- If such an L exists, it is unique.

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Polar coordinates to the rescue! $x = r\cos\theta$, $y = r\sin\theta$ and

$$(x,y) \to (0,0) \iff r \to 0$$

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(Squeeze Theorem in action!)

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$$f(x,y) = \frac{xy}{x^2+y^2} \Longrightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = ?$$

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Example: $\mathbf{u} = \langle 1, m\rangle \implies x = t,\ y = mt \implies y = mx$

$$\lim_{t\to 0} f(t\mathbf{u}) = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

depends on m (hence on $\mathbf{u}$).

Side Limits and Directional Limits

Similarity:

- Single variable functions:
  Limit exists $\Longrightarrow$ side limits exist, have the same value.
  Conclusion: Side limits are different $\Longrightarrow$ limit does not exist.
- Multivariable functions:
  Limit exists $\Longrightarrow$ directional limits exist, have the same value.
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Difference:

- Single variable functions:
  Side limits are equal $\Longrightarrow$ limit exists.
- Multivariable functions:
  All directional limits have the same value $\Longrightarrow$ limit does not necessarily exist.

A trickier example

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

Then along $y = mx$:

$$\frac{xy^2}{x^2+y^4} = \frac{mx^3}{x^2+m^4x^4} = \frac{mx}{1+m^2x^4} \to 0 \text{ as } x \to 0$$

for all m.

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Conclusion:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} \quad \text{does not exist}$$

Limits along paths

$$f \colon D \to \mathbb{R} \quad , \quad \mathbf{r} \colon I \to \mathbb{R}^2$$

such that

- 0 is in I, $\mathbf{r}(0) = \mathbf{r}_0$;
- $\mathbf{r}$ is continuous at 0;
- $\mathbf{r}(t)$ is in D for $t \neq 0$

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Particular case: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$ (Directional limit)
Then:

$$\lim_{Q \to P_0} f(Q) = L \Longleftrightarrow \lim_{t \to 0} f(\mathbf{r}(t)) = L$$

for all continuous paths $\mathbf{r}$ such that $\mathbf{r}(0) = \mathbf{r}_0$.

Continuity

- f is continuous at (x_0, y_0) if

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- Sum, difference, product, quotient of continuous functions are continuous;
- Powers, exponentials of continuous functions are continuous;
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What can go wrong:

- Limit different from actual value: Removable discontinuity:
- Limit does not exist: essential discontinuity.

Continuity of vector fields

Vector field:

$$\mathbf{F}\colon D \to \mathbb{R}^2, \mathbf{F}(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$$

$$F_1, F_2\colon D \to \mathbb{R} \implies \text{ scalar output.}$$

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Example:

$$\mathbf{F}(x,y) = y\mathbf{i} - x\mathbf{j}$$

$$F_1(x,y) = y \quad , \quad F_2(x,y) = -x$$

Polynomials $\implies$ continuous components $\implies$ continuous field