# Derivatives

March 1, 2010

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$$\lim_{P \to P_0} \frac{f(P) - f(P_0)}{|P_0 P|}$$

does not exist!

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# Directional Derivatives

$f: D \to \mathbb{R}$, $P_0(\mathbf{r}_0)$ in $D$, $\mathbf{u}$ nonzero vector
$L$: line through $P_0$ with direction $\mathbf{u}$, oriented by $\mathbf{u}$.

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$(D_{\mathbf{u}} f)(P_0) =$ instantaneous rate of change of $f$ with respect to change in *position* along the line with unit direction $\mathbf{u}$.

# Partial Derivatives

$Oxy$: rectangular coordinate system in the plane
$f : \mathcal{R} \to \mathbb{R}$, $P_0(x_0, y_0)$ inside $\mathcal{R}$

$$\mathbf{u} = \mathbf{i} \Longrightarrow \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u} = \langle x_0 + t, y_0 \rangle$$

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Notations for partial derivatives:

$$(D_{\mathbf{i}} f)(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0)$$

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We compute partial derivatives by

- keeping all other variables constant and
- applying the rules for differentiation for single variable functions.

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$$f_x = \frac{\partial f}{\partial x} = \frac{\partial (y^2 \ln(2x+y) - e^y)}{\partial x} = \frac{\partial (y^2 \ln(2x+y))}{\partial x} - \frac{\partial (e^y)}{\partial x} =$$

$$= y^2 \frac{\partial (\ln(2x+y))}{\partial x} - 0 = y^2 \cdot \frac{1}{2x+y} \cdot \frac{\partial (2x+y)}{x} = \frac{2y^2}{2x+y} .$$

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$$= y^2 \cdot \frac{1}{2x+y} \cdot \frac{\partial(2x+y)}{y} + 2y \ln(2x+y) - e^y =$$

$$= \frac{2y^2}{2x+y} + 2y \ln(2x+y) - e^y.$$

# Graphical Interpretation

The graph of $f$ is the surface

$$(x, y) \rightarrow \langle x, y, f(x, y) \rangle$$

The vertical plane containing the line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{i}$ is the plane $y = y_0$.
Intersection of graph with the plane $y = y_0$ is the curve

$$\gamma(t) = \langle t, y_0, f(t, y_0) \rangle \implies \text{Graph of } z = h(x)$$

Direction of tangent line to $\gamma$:

$$\gamma'(x_0) = \langle 1, 0, f_x(x_0, y_0) \rangle$$

In the $xz-$plane $y = y_0$, the slope of this line is

$$h'(x_0) = f_x(x_0, y_0)$$

# Higher Order Derivatives

Partial derivative functions:

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The partial derivatives of the partial derivatives $f_x$ and $f_y$ are called *the second order partial derivatives* of $f$.
The partial derivatives of the second order derivatives are the *third order* derivatives, and so on.

$$f(x,y) \to \begin{cases} \frac{\partial f}{\partial x} = f_x \to & \begin{cases} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = & \frac{\partial^2 f}{\partial x^2} = & f_{xx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = & \frac{\partial^2 f}{\partial y\,\partial x} = & f_{xy} \end{cases} \\ \frac{\partial f}{\partial x} = f_y \to & \begin{cases} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = & \frac{\partial^2 f}{\partial x\,\partial y} = & f_{yx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = & \frac{\partial^2 f}{\partial y^2} = & f_{yy} \end{cases} \end{cases}$$

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$$f_x(x, y) = 2xy^3 \qquad f_y(x, y) = 3x^2y^2$$

$$f_{xx}(x, y) = (2xy^3)_x = 2y^3 \qquad f_{xy}(x, y) = (2xy^3)_y = 6xy^2$$

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$$\begin{aligned} f_x(x,y) &= 2xy^3 & f_y(x,y) &= 3x^2y^2 \\ f_{xx}(x,y) &= (2xy^3)_x = 2y^3 & f_{xy}(x,y) &= (2xy^3)_y = 6xy^2 \\ f_{yx}(x,y) &= (3x^2y^2)_x = 6xy^2 & f_{yy}(x,y) &= (3x^2y^2)_y = 6x^2y \end{aligned}$$

Notice that $f_{xy} = f_{yx}$. That is not a coincidence.

## Theorem (Clairaut)

*If the second order derivatives $f_{xy}$ and $f_{yx}$ are continuous on an open domain (such as an open disk, for example), then they are equal everywhere on the domain.*

Similar results for functions of three variables.