Linear Approximations

March 3, 2010

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Intuitively: It should include all vectors tangent at P to curves passing through P and contained in the surface.

Equivalently: It should be the geometric plane

- passing through P;
- parallel to the directions of all tangent vectors of curves passing through P and contained in the surface.

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$$x \to \langle x, y_0, f(x, y_0)\rangle \implies \text{ tangent direction } \langle 1, 0, f_x(x_0, y_0)\rangle$$
$$y \to \langle x_0, y, f(x_0, y)\rangle \implies \text{ tangent direction } \langle 0, 1, f_y(x_0, y_0)\rangle\ .$$

Vector normal to the tangent plane:

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Vector normal to the tangent plane:

$$\mathbf{n} = \langle 1, 0, f_x(x_0, y_0)\rangle \times \langle 0, 1, f_y(x_0, y_0)\rangle = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle \ .$$

Equation of the tangent plane at $P_0(x_0, y_0, z_0 = f(x_0, y_0))$:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \iff$$

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - f(x_0, y_0)) = 0 \iff$$
$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)} \ .$$

Linearizations

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}\,.$$

$$L_{f,(x_0,y_0)}(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

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at the point corresponding to $(x, y) = (4, 1)$.
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Partial derivatives at $(4, 1)$:

$$f_x(x, y) = 2x + y \Longrightarrow f_x(4, 1) = 9$$
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$$z = 22 + 9(x - 4) + 8(y - 1) \iff z = 9x + 8y - 22$$

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Good Linear Approximations

The linearization provides an *approximation* of a function around a point.
Question: How does the error in output, $|f(x,y) - L_{f,(x_0,y_0)}|$, compare to the error in input, $\sqrt{(x-x_0)^2 + (y-y_0)^2}$, for small errors in input?

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In our example, using $x = 4 + r\cos\theta$, $y = 1 + r\sin\theta$:

$$\lim_{(x,y)\to(4,1)} \frac{|x^2 + xy + 2y^2 - (9x + 8y - 22)|}{\sqrt{(x-4)^2 + (y-1)^2}} = 0$$

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Conclusion: $P(x, y) = 9x + 8y - 22$ is a good linear approximation for $f(x, y) = x^2 + xy + 2y^2$ around $(4, 1)$.

Differentiability

If $y = h(x)$ is a function of one variable, then

$$L_{h,x_0}(x) = h(x_0) + h'(x_0)(x - x_0)$$

$$\lim_{x \to x_0} \frac{|h(x) - L_{h,x_0}(x)|}{|x - x_0|} = \lim_{x \to x_0} \left| \frac{h(x) - h(x_0)}{x - x_0} - h'(x_0) \right| = 0$$

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Fact: If f has a good linear approximation at (x_0, y_0), then the good linear approximation is necessarily the linearization of f at (x_0, y_0).

Continuity and Differentiability

Single variable functions:
differentiable at a point if and only if it has a derivative at that point.

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- If f is differentiable at (x_0, y_0), then:
  - f is continuous at (x_0, y_0);
  - f has partial derivatives at (x_0, y_0).
- If f has partial derivatives at (x_0, y_0) then
  - f need not be differentiable at (x_0, y_0);
  - f need not be even continuous at (x_0, y_0).

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Source of these problems:

- Partial derivatives at a point take into consideration the behavior of the function along *special directions*;
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Consequences:

- Polynomial functions are differentiable;
- Algebraic combinations of differentiable functions are differentiable.

Total Differential

If f is differentiable at (x_0, y_0), then

$$f(x,y) \simeq f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$\Delta f \simeq f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

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For infinitesimally small Δx and Δy we get:
Definition: The total differential df at (x_0, y_0) is

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Alternatively:

$$df = f_x dx + f_y dy \qquad \text{or} \qquad df = f_x dx + f_y dy + f_z dz$$

$$\begin{aligned} \Delta f &: \text{actual change in } f \\ df \simeq \Delta f &: \text{infinitesimal change in } f \\ f_x(x_0, y_0), f_y(x_0, y_0) &: \text{error propagation factors} \end{aligned}$$

Application

A cylinder has radius $r = 3cm$ and height $h = 5cm$. The error in measuring the radius is $\pm 1mm$, and the error in measuring the height is $\pm 1mm$. Estimate the error in the volume of the cylinder.

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The error in volume, ΔV, is estimated by dV:

$$\Delta V \simeq dV = V_r(3,5)dr + V_h(3,5)dh \simeq V_r(3,5)\Delta r + V_h(3,5)\Delta h\,.$$

$$V_r(r,h) = 2\pi rh \Longrightarrow V_r(3,5) = 30\pi$$

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$$\Delta V \simeq (30\pi)(\pm 0.1) + ((9\pi)(\pm 0.1) \Longrightarrow V(r,h) \simeq V(3,5) \pm 3.9\pi\, cm^3$$

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Remark: Since $V_r(3,5) > V_h(3,5)$, the result is more sensitive to errors in r than to errors in h.