Chain Rule

March 3, 2010

Motivation

Recall:

- f, differentiable function,
- $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, unit vector,
- $P(x_0, y_0, z_0)$, point.

What is the rate of change of f at P in the direction $\mathbf{u}$?

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Directional derivative

$$(D_{\mathbf{u}} f)(P) = \left. \frac{d}{dt} \right|_{t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

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More general, if

- $w = w(x, y, z)$;
- $x = x(t)$, $y = y(t)$, $z = z(t)$,

and all the functions are differentiable, how do we compute $\frac{dw}{dt}$?

Chain Rule

Differentials

$$dw = w_x(x, y, z)dx + w_y(x, y, z)dy + w_z(x, y, z)dz$$

and

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Then

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Derivative of composition of functions $\Longrightarrow$ Chain Rule

Tree Diagrams

- $w = w(x, y, z)$;
- $x = x(t)$, $y = y(t)$, $z = z(t)$,

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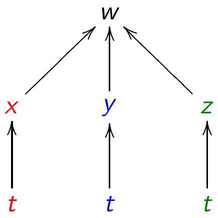
Alternative way of arranging terms - tree diagram:



(More) General Chain Rule

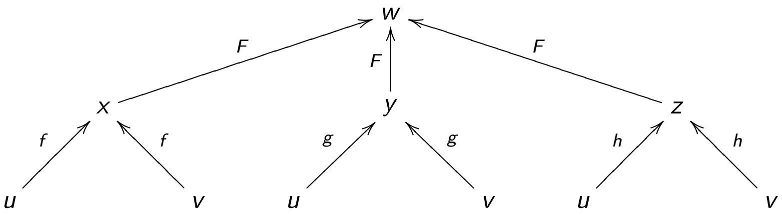
More general formula:

- $w = F(x, y, z)$;
- $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$.

$$w = F(f(u, v), g(u, v), h(u, v)) = G(u, v)$$

To compute $\frac{\partial w}{\partial u} = \frac{\partial G}{\partial u}$:

- arrange variables in a tree diagram:



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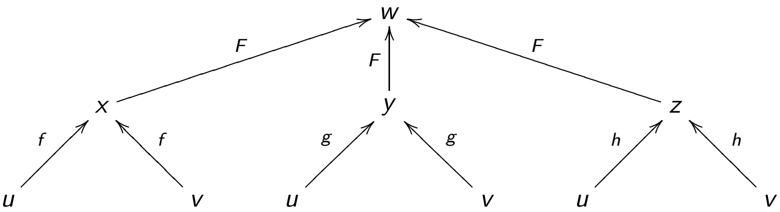

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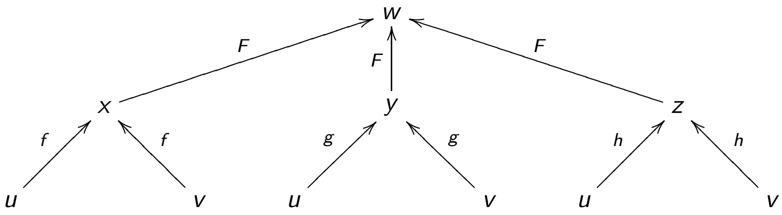

- Think of each edge $a \xrightarrow{H} b$ as having a label $\frac{\partial b}{\partial a} = \frac{\partial H}{\partial a}$;
- Identify all the end vertices that are labeled u;
- Identify all the paths connecting an end vertex u to the root w. In our case there are three such paths:
  - $u \longrightarrow x \longrightarrow w$;
  - $u \longrightarrow y \longrightarrow w$;
  - $u \longrightarrow z \longrightarrow w$.

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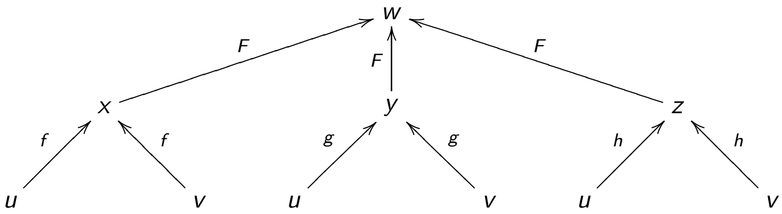

For each path add a term consisting of product of (partial) derivatives along edges. In our case:

First path contribution: $\frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} = \frac{\partial F}{\partial x} \cdot \frac{\partial f}{\partial u}$

Second path contribution: $\frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial F}{\partial y} \cdot \frac{\partial g}{\partial u}$

Third path contribution: $\frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} = \frac{\partial F}{\partial z} \cdot \frac{\partial h}{\partial u}$

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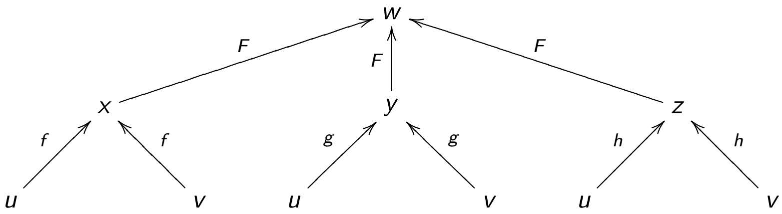

The derivative is the sum of the contributions along paths:

$$\frac{\partial w}{\partial u} = \frac{\partial G}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} =$$

$$= \frac{\partial F}{\partial x}\cdot\frac{\partial f}{\partial u} + \frac{\partial F}{\partial y}\cdot\frac{\partial g}{\partial u} + \frac{\partial F}{\partial z}\cdot\frac{\partial h}{\partial u}$$

Example: powerexponential

Let $f(x) = x^x$. Compute $f'(x)$.

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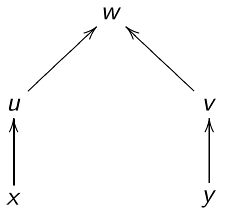

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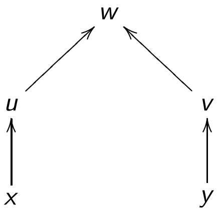


Then $f(x) = w(u(x), v(x))$ and

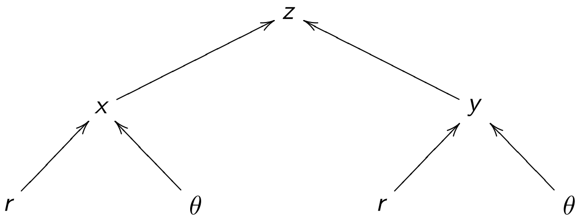
$$f'(x) = \frac{\partial w}{\partial u}\frac{du}{dx} + \frac{\partial w}{\partial v}\frac{dv}{dx} = vu^{v-1} + u^v \ln u = x \cdot x^{x-1} + x^x \ln x = x^x(1 + \ln x) .$$

Example: Derivatives in polar coordinates

Let $z = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$. Then

$$z = f(r\cos\theta, r\sin\theta) = g(r, \theta)\ .$$

$P(x, y) = P(r, \theta)$. Compute $\frac{\partial z}{\partial r} = \frac{\partial g}{\partial r}$ at P:



Then:

$$\frac{\partial z}{\partial r}(P) = \frac{\partial z}{\partial x}(P)\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}(P)\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(P)\cos\theta + \frac{\partial z}{\partial y}(P)\sin\theta$$

hence

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

Partial Derivatives in Polar Coordinates

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y} \; .$$

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Solving the linear system we get:

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}$$

The Laplace Operator

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$$z \to \Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \Longrightarrow \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

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In polar coordinates:

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

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$$f(x, y) = g(r, \theta) = \ln(r^2) = 2 \ln r \ .$$

Then

$$\begin{aligned}\Delta f = \Delta g =& \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial (2\ln r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 (2\ln r)}{\partial \theta^2} \\ =& \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{2}{r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(2) = 0 \ .\end{aligned}$$

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Fact: The only harmonic functions independent of θ are of the form

$$g(r,\theta) = c_1\ln r + c_2\ .$$