Directional Derivatives

March 8, 2010

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- f, differentiable function,
- $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, unit vector,
- $P(x_0, y_0, z_0)$, point.

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What is the rate of change of f at P in the direction $\mathbf{u}$?
Directional derivative

$$(D_{\mathbf{u}} f)(P) = \frac{d}{dt}\bigg|_{t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

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$$\mathbf{W}_{f,(x_0,y_0,z_0)} = \langle \frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0)\rangle$$

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Then

$$(D_{\mathbf{u}}f)(x_0, y_0, z_0) = \left.\frac{dw}{dt}\right|_{t=0} = \mathbf{W}_{f,(x_0,y_0,z_0)} \cdot \mathbf{u} \ .$$

Example

Directional derivative

- of function $f(x, y, z) = \ln\left(x^2 + 2y^2 - z^2\right)$;
- at the point $P(2, 1, -1)$;
- in the direction $\mathbf{v} = \langle -1, 2, 1 \rangle$.

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A unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{6}}\langle -1, 2, 1\rangle \ .$$

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The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + 2y^2 - z^2} \implies \frac{\partial f}{\partial x}(2, 1, -1) = \frac{4}{5}$$
$$\frac{\partial f}{\partial y} = \frac{4y}{x^2 + 2y^2 - z^2} \implies \frac{\partial f}{\partial y}(2, 1, -1) = \frac{4}{5}$$
$$\frac{\partial f}{\partial z} = \frac{-2z}{x^2 + 2y^2 - z^2} \implies \frac{\partial f}{\partial z}(2, 1, -1) = \frac{2}{5}$$

$$\mathbf{W}_{f,(2,1,-1)} = \langle \frac{4}{5}, \frac{4}{5}, \frac{2}{5} \rangle$$

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$$(D_{\mathbf{u}}f)(2, 1, -1) = \mathbf{W}_{f,(2,1,-1)} \cdot \mathbf{v} = \frac{\sqrt{6}}{5}$$

$(D_{\mathbf{u}}f)(2, 1, -1) > 0 \implies$ if we start at $(2, 1, -1)$ and move in the direction **u**, then f is increasing.

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Fact: If the maximal rate of increase is strictly positive, then it is achieved in exactly one direction. The answers are encoded in a vector.

Definition

The *gradient vector* of f at P is the unique vector that has

- magnitude equal to the maximal rate of increase of f from P.
- If the magnitude is not zero, then the direction is the direction in which f increases the fastest from P.

Notation: $(\nabla f)(P) = (\nabla f)_P = (\textbf{grad } f)(P)$

- ∇f: gradient vector field $\qquad P \rightsquigarrow (\nabla f)(P)$
- ∇: *del* operator $\qquad$ function $f \rightsquigarrow$ vector field ∇f

Coordinate Computation

If **u** is a unit vector then:

$$(D_{\mathbf{u}} f)(x_0, y_0, z_0) = \mathbf{W}_{f,(x_0,y_0,z_0)} \cdot \mathbf{u} = \left|\mathbf{W}_{f,(x_0,y_0,z_0)}\right| \cos\alpha \ .$$

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If $|\mathbf{W}_{f,(x_0,y_0,z_0)}| \neq 0$, then

- $(D_{\mathbf{u}}f)(x_0, y_0, z_0)$ is maximal when $\alpha = 0$, hence if

$$\mathbf{u} = \frac{1}{|\mathbf{W}_{f,(x_0,y_0,z_0)}|} \mathbf{W}_{f,(x_0,y_0,z_0)}$$

- The maximal value of $(D_{\mathbf{u}}f)(x_0, y_0, z_0)$ is $|\mathbf{W}_{f,(x_0,y_0,z_0)}|$.

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Therefore, in rectangular coordinates, we have

$$\begin{aligned}(\nabla f)(x_0, y_0, z_0) =& |\mathbf{W}_{f,(x_0,y_0,z_0)}|\mathbf{u} = \mathbf{W}_{f,(x_0,y_0,z_0)} = \\ =& \langle \frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0) \rangle\end{aligned}$$

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Coordinate-free formula for directional derivatives:

$$(D_{\mathbf{u}}f)(P) = (\nabla f)_P \cdot \mathbf{u}$$

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Using the chain rule we get

$$(\nabla_{\gamma'(t_0)} f)(\gamma(t_0)) = \left.\frac{d}{dt}\right|_{t=t_0} f(\gamma(t)) = (\nabla f)_{\gamma(t_0)} \cdot \gamma'(t_0)\ .$$

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If $\mathbf{u}$ is a unit vector, $\gamma(t_0) = P$ and $\gamma'(t_0) = \mathbf{u}$, then:

$$(D_{\mathbf{u}} f)(P) = (\nabla f)_P \cdot \mathbf{u} = (\nabla f)_{\gamma(t_0)} \cdot \gamma'(t_0) = \left.\frac{d}{dt}\right|_{t=t_0} f(\gamma(t)) \, .$$

Gradient in Polar Coordinates

$\mathbf{e}_r = \mathbf{e}_r(P)$ and $\mathbf{e}_\theta = \mathbf{e}_\theta(P)$ are the polar fundamental directions at P

$$(\nabla f)_P = a\mathbf{e}_r + b\mathbf{e}_\theta$$

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$\mathbf{e}_r$ and $\mathbf{e}_\theta$ perpendicular unit vectors $\Longrightarrow$

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To compute $(D_{\mathbf{e}_r} f)(P)$ we use the line through $P(r_0, \theta_0)$ with direction $\mathbf{e}_r$, which in polar coordinates is given by $(r, \theta) = (t, \theta_0)$. Therefore

$$a = (D_{\mathbf{e}_r} f)(P) = \left.\frac{d}{dt}\right|_{t=r_0} f(t, \theta_0) = \frac{\partial f}{\partial r}(P)\ .$$

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To compute $(D_{\mathbf{e}_\theta} f)(P)$ we use the circle centered at the origin and passing through $P(r_0, \theta_0)$. The polar parametrization of this circle that has *unit* tangent at P is given by $(r, \theta) = (r_0, \frac{1}{r_0} t)$. Therefore

$$b = (D_{\mathbf{e}_\theta} f)(P) = \left.\frac{d}{dt}\right|_{t=\theta_0} f\left(r_0, \frac{1}{r_0} t\right) = \frac{1}{r_0} \frac{\partial f}{\partial \theta}(P) \,.$$

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From the previous computations:

$$(\nabla f)_P = \frac{\partial f}{\partial r}(P)\mathbf{e}_r + \frac{1}{r_0}\frac{\partial f}{\partial \theta}(P)\mathbf{e}_\theta \ ,$$

or

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \ .$$

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Problem: Let $\mathbf{X}$ be a vector field of the form

$$\mathbf{X} = h(r)\mathbf{r}$$

for some continuous function h. Show that $\mathbf{X}$ is a *gradient field*: there exists a smooth function f such that $\mathbf{X} = \nabla f$.

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<u>Question</u>: Is the object moving in that direction?