Implicit Functions

March 10, 2010

Tangent Plane to Level Surface

Given:

- a differentiable function F,
- a level surface $S = \{Q \mid F(Q) = k\}$,
- point P on this level set

Question:

What should be the tangent plane/line to S at P?

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Intuitive answer:
Analogy to graph surfaces: the tangent plane should contain tangent vectors at P to curves through P and contained in S.

Regular Points

Let $\mathbf{r} = \mathbf{r}(t)$ be a curve that

- passes through P for $t = 0$: $\mathbf{r}(0) = \mathbf{r}_0$, the position vector of P;
- is contained in the level set S: $F(\mathbf{r}(t)) \equiv k$ for all t.

Then

$$\frac{d(F(\mathbf{r}(t))}{dt} \equiv 0 \Longrightarrow (\nabla F)(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \equiv 0 \Longrightarrow (\nabla F)_P \cdot \mathbf{r}'(0) = 0$$

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<u>Definition</u>: A point P is called

- a *regular* point for F if $(\nabla F)_P \neq \mathbf{0}$;
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<u>Fact</u>: At a regular point, the gradient is orthogonal to the level set through that point.

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- Function: $F = F(x, y, z)$;
- Point: $P = P(x_0, y_0, z_0)$, hence $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$;
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- Gradient of F at P:

$$(\nabla F)_P = \langle \frac{\partial F}{\partial x}(x_0, y_0, z_0), \frac{\partial F}{\partial y}(x_0, y_0, z_0), \frac{\partial F}{\partial z}(x_0, y_0, z_0) \rangle$$

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- Equation of tangent plane to S at P:

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Example

If

$$F(x, y, z) = x^2 + y^2 - z^2 \ ,$$

then the level surface through $P(1, 2, 3)$ is the surface S with equation

$$x^2 + y^2 - z^2 = F(1, 2, 3) \iff x^2 + y^2 - z^2 = -4 \ .$$

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$$\langle 2, 4, -6 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle = 0 \Longrightarrow$$

$$2(x - 1) + 4(y - 2) - 6(z - 3) = 0 \iff x + 2y - 3z = -4$$

Tangent Planes to Quadric Surfaces

Let $F(x, y, z) = Ax^2 + By^2 + Cz^2$ such that not all of A, B, C are zero.

<u>Fact</u>: If $Q(a, b, c)$ is a critical point, then

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Suppose $k \neq 0$. Let $P(x_0, y_0, z_0)$ be a point on S. Then

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$$(\nabla F)(x_0, y_0, z_0) = \langle 2Ax_0, 2By_0, 2Cz_0 \rangle .$$

- The equation of the tangent plane is

$$2Ax_0(x - x_0) + 2By_0(y - y_0) + 2Cz_0(z - z_0) = 0 \Longrightarrow Ax_0 x + By_0 y + Cz_0 z = k$$

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A graph surface $z = f(x, y)$ can always be represented as a level surface:

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  Think $x^2 + y^2 + z^2 = 1$.
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  Around $P(0, 0, 1)$, the surface is the graph surface of $z = f(x, y)$ with $f(x, y) = \sqrt{1 - x^2 - y^2}$.

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How about the level surface

$$x^3 + y^3 + z^3 + 6xyz = -4$$

around the point $(1, 2, -1)$?

Implicit Functions

Given:

- a function F,
- a point $P(x_0, y_0, z_0)$ in the domain of F,
- the level surface S of F through P:

$$F(x, y, z) = k \qquad \text{with } k = F(x_0, y_0, z_0)$$

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S is a graph surface around P if there is a function $z = f(x, y)$ such that:

- f is defined on an open disk D around (x_0, y_0);
- $f(x_0, y_0) = z_0$;
- $F(x, y, f(x, y)) = k$ for all (x, y) in the disk D.

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The equation $F(x, y, z) = k$ *implicitly* defines the function f satisfying

- $F(x, y, f(x, y)) = k$ for all (x, y) in the disk D.
- $f(x_0, y_0) = z_0$.

Examples

The equation $x^2 + y^2 + z^2 = 1$ implicitly defines $z = \sqrt{1 - x^2 - y^2}$ as the unique function $z = f(x, y)$ such that

- $x^2 + y^2 + (f(x, y))^2 = 1$ for all (x, y) in a disk around $(0, 0)$;
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In general, the implicit function question is quite tricky!
But for differentiable functions things are somehow simpler.

Implicit Function Theorem

- F: function
  - defined for all points in a ball around the point $P(x_0, y_0, z_0)$,
  - with continuous partial derivatives.
- S: the level surface of F through P, for the level $k = F(x_0, y_0, z_0)$.

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If $F_z(x_0, y_0, z_0) \neq 0$, then there exists a function $z = f(x, y)$

- defined on an open disk D centered at (x_0, y_0),
- such that $F(x, y, f(x, y)) = k$ for all (x, y) in D,
- and such that $f(x_0, y_0) = z_0$.

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Moreover:

- For each fixed disk D the function $z = f(x, y)$ is unique.
- The function $z = f(x, y)$ has partial derivatives and

$$z_x(x, y) = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad z_y(x, y) = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

Consequences

If $F_z(x_0, y_0, z_0) \neq 0$, then the level surface S

$$F(x, y, z) = F(x_0, y_0, z_0)$$

can be locally represented as a graph surface

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around the point (x_0, y_0, z_0).

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around the point (x_0, y_0, z_0). Nothing special about z: if

$$(\nabla F)(x_0, y_0, z_0) = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\rangle \neq \mathbf{0} \ ,$$

then the level surface S can be represented as a graph surface

- $z = f(x, y)$, if $F_x(x_0, y_0, z_0) \neq 0$;
- $y = g(x, z)$, if $F_y(x_0, y_0, z_0) \neq 0$;
- $x = h(y, z)$, if $F_z(x_0, y_0, z_0) \neq 0$;

Implicit Differentiation

If

$$F(x, y, f(x, y)) = 0$$

for all (x, y) in an open disk, then, by differentiating with respect to x and using the chain rule, we get:

$$F_x(x, y, z) + F_z(x, y, z) f_x(x, y) = 0$$

Since $F_z(x, y, z)$ is continuous

$$F_z(x_0, y_0, z_0) \neq 0 \Longrightarrow F_z(x, y, z) \neq 0$$

for all (x, y, z) in an open ball around (x_0, y_0, z_0). For such points we get

$$\frac{\partial z}{\partial x} = f_x(x, y) = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \ ,$$

and similarly for the partial derivative with respect to y.

Example

Show that the equation $x^3 + y^3 + z^3 + 6xyz = -4$ implicitly defines a function $z = f(x, y)$ around the point $(1, 2)$, such that $f(1, 2) = -1$. Compute the partial derivatives of this function at $(1, 2)$.

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Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz$. Then

$$F_z(x, y, z) = 3z^2 + 6xy \Longrightarrow F_z(1, 2, -1) = 15$$

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Since $F_z(1, 2, -1) \neq 0$, it follows that $F(x, y, z) = F(1, 2, -1) = 1$ defines z implicitly in terms of (x, y) around $(1, 2)$, and

$$z_x(x, y) = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} \Longrightarrow z_x(1, 2) = \frac{3}{5}\ .$$

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$$z_x(x, y) = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} \Longrightarrow z_x(1, 2) = \frac{3}{5} .$$

Similarly

$$z_y(x, y) = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} \Longrightarrow z_y(1, 2) = -\frac{2}{5} .$$

Tangent Plane

Equation of tangent plane to

$$x^3 + y^3 + z^3 + 6xyz = -4$$

at $(1, 2, -1)$.

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- As level surface of $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz$:

$$F_x(1, 2, -1)(x - 1) + F_y(1, 2, -1)(y - 2) + F_z(1, 2, -1)(z + 1) = 0$$

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- As graph of $z = f(x, y)$:

$$z = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2)$$

$$z = -1 + \frac{3}{5}(x - 1) - \frac{2}{5}(y - 2)$$

Intuitive Argument

Actual equation:

$$F(x, y, z) = F(x_0, y_0, z_0) \Longrightarrow z = f(x, y)$$

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Linearized equation:

$$L_{F,(x_0,y_0,z_0)}(x, y, z) = F(x_0, y_0, z_0) \Longrightarrow z = L_{f,(x_0,y_0)}(x, y)$$

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$F_z(x_0, y_0, z_0) \neq 0 \Longrightarrow$ *can* solve $L_{F,(x_0,y_0,z_0)}(x, y, z) = F(x_0, y_0, z_0)$ for z:

$$z = z_0 - \frac{F_x(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}(x - x_0) - \frac{F_y(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}(y - y_0) = L_{f,(x_0,y_0)}(x, y)$$

Then

$$z_x(x_0, y_0) = -\frac{F_x(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)} \quad \text{and} \quad z_y(x_0, y_0) = -\frac{F_y(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}.$$