Parametrized Surfaces

March 22, 2010

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Surface: A two dimensional object in space.

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Graph surfaces:

Let $D \subset \mathbb{R}^2$ be an open set and $f : D \to \mathbb{R}$ a continuous function.

Then

$$S = \{(x, y, z) \mid (x, y) \in D, z = f(x, y)\}$$

is a surface: for every point (x_0, y_0, z_0) on S, there exists an open disk B centered at (x_0, y_0) and included in D. Then the patch

$$E = \{(x, y, z) \mid (x, y) \in B, z = f(x, y)\}$$

is an open patch on S around P and f identifies E with B.

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Planes:

S: plane through P_0 and parallel to directions $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v} \ ,$$

Function $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^3$,

$$\mathbf{F}(s,t) = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$$

globally identifies S with the plane $\mathbb{R}^2$.

- $S \ni P \Longleftrightarrow$ pair of coordinates (s,t);
- The vectors $\mathbf{F}_s = \mathbf{u}$ and $\mathbf{F}_t = \mathbf{v}$ are non-collinear.

Local parametrizations

A *(differentiable) local parametrization*: function $\mathbf{F}\colon D \to \mathbb{R}^3$,

- defined on an open subset $D \subset \mathbb{R}^2$;
- given by
$$\mathbf{F}(s,t) = (f_1(s,t), f_2(s,t), f_3(s,t))$$

such that

- $\mathbf{F}$ is 1-1: each point in the image has a unique pair of coordinates;
- The components f_1, f_2, f_3 are differentiable functions from D to $\mathbb{R}$.
- The vectors $\mathbf{F}_s$ and $\mathbf{F}_t$ are non-collinear.

Image of $\mathbf{F}$: surface in $\mathbb{R}^3$.

Example

Let R be fixed and $\mathbf{F}\colon (0,\pi)\times(0,2\pi)\to\mathbb{R}^3$ be given by

$$\mathbf{F}(s,t) = \langle R\sin s\cos t, R\sin s\sin t, R\cos s\rangle\ .$$

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- $\mathbf{F}_s$ and $\mathbf{F}_t$ are non-collinear:

$$\mathbf{F}_s = \langle R\cos s\cos t, R\cos s\sin t, -R\sin s\rangle$$
$$\mathbf{F}_t = \langle -R\sin s\sin t, R\sin s\cos t, 0\rangle$$

$$|\mathbf{F}_s\times\mathbf{F}_t| = R^2\sin s\neq 0$$

Parametrized surfaces

A subset S in space is a *smooth surface* if

- For every point P of S, there is a smooth local parametrization of an open patch on S around P;
- When images of two local parametrizations overlap on S, the two local parametrizations should be compatible.

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- The sphere $x^2 + y^2 + z^2 = R^2$ is a surface.

Non-example:
The cone $z^2 = x^2 + y^2$ is not globally a surface, because there is no patch around $(0, 0, 0)$ that looks like an open set in the plane.

Level Surfaces as Parametrized Surfaces

Let $f: \mathbb{R}^3 \to R$ be a differentiable function.

- P is a *critical point* if $(\nabla f)(P) = 0$;
- $c \in \mathbb{R}$ is a *critical value* if
$$f^{-1}(c) = \{(x, y, z) \mid f(x, y, z) = c\}$$
contains a critical point.
- If c is not a critical value then it is a *regular* value.

Implicit Function Theorem $\implies$
If c is a regular value taken by f, then $f^{-1}(c)$ is a smooth surface.
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Examples:
For $f(x, y, z) = ax + by + cz$ with a, b, c not all zero,

- All points P are regular points
- All values d are regular;
- $f^{-1}(d)$ is a smooth surface: the plane $ax + by + cz = d$.

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<u>Examples</u>:
For $f(x, y, z) = x^2 + y^2 - z^2$,

- $(0, 0, 0)$ is the only critical point $\Longrightarrow$ $0 = f(0, 0, 0)$ is the only critical value.
- $H = f^{-1}(1)$ is a smooth surface: $x^2 + y^2 - z^2 = 1$, a hyperboloid with one sheet.

Surfaces of Revolution

- C be a curve in the $(x, z)-$plane;
- S be the surface obtained by revolving C around the $z-$axis.

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$$|\mathbf{F}_u \times \mathbf{F}_v| = |f|\sqrt{(f')^2 + (g')^2}\,,$$

hence $\mathbf{F}$ is a smooth parametrization if $f(u) \neq 0$ for all u.

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Geometrically: the curve C shouldn't intersect the $z-$axis.

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Let C be the circle

- in the $(x, z)-$plane,
- of radius r,
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The image of F is S minus two circles:

- the vertical circle $x^2 + z^2 = 1, y = 0$ $(u \neq 0)$
- the horizontal circle $x^2 + y^2 = (R + r)^2, z = 0$ $(v \neq 0)$.

Tangent Plane

Let $\mathbf{F}\colon D \to \mathbb{R}^3$, $\mathbf{F}(u,v) = (f(u,v), g(u,v), h(u,v))$ be a local parametrization of a surface S,
$P = \mathbf{F}(u_0, v_0)$: point on S.
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We get curves on S from curves in D, using $\mathbf{F}$:

$$\gamma(t) = \mathbf{F}(u(t), v(t)) = (f(u(t), v(t)), g(u(t), v(t)), h(u(t), v(t)))$$

$$\gamma(0) = P \implies u(0) = u_0 \text{ and } v(0) = v_0$$

$$\gamma'(0) = u'(0)\mathbf{F}_u(u_0, v_0) + v'(0)\mathbf{F}_v(u_0, v_0)$$

Every tangent vector at P is a linear combination of $\mathbf{F}_u$ and $\mathbf{F}_v$.

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Tangent plane: parallel to directions $\mathbf{F}_u(u_0, v_0)$ and $\mathbf{F}_v(u_0, v_0)$.
Normal to the plane tangent to S at $P = \mathbf{F}(u_0, v_0)$:

$$\mathbf{N}(u_0, v_0) = \mathbf{F}_u(u_0, v_0) \times \mathbf{F}_v(u_0, v_0)$$

Equation for the tangent plane:

$$(\mathbf{r} - \mathbf{F}(u_0, v_0)) \cdot \mathbf{N}(u_0, v_0) = 0 \; .$$

A vectorial parametric equation for the line normal to S at P is

$$\mathbf{r}(t) = \mathbf{F}(u_0, v_0) + t\mathbf{N}(u_0, v_0) \; .$$

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$$\mathbf{N}(1, \pi/3) = \mathbf{F}_u(1, \pi/3) \times \mathbf{F}_v(1, \pi/3) \Longrightarrow \mathbf{N}(1, \pi/3) = -2\,\mathbf{i} - 2\sqrt{3}\,\mathbf{j} + 2\,\mathbf{k}$$

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$$(\langle x, y, z\rangle - \langle 1, \sqrt{3}, 1\rangle) \cdot \langle -2, -2\sqrt{3}, 2\rangle = 0$$

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Example

Let $\mathbf{F}(u,v) = (u\cos v, u\sin v, e^{2u-4})$ and $P = \mathbf{F}(2,\pi/3)$.

$$\mathbf{F}_u(u,v) = \langle \cos v, \sin v, 2e^{2u-4}\rangle \Longrightarrow \mathbf{F}_u(1,\pi/3) = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 2\rangle$$

$$\mathbf{F}_v(u,v) = \langle -u\sin v, u\cos v, 0\rangle \Longrightarrow \mathbf{F}_v(1,\pi/3) = \langle -\sqrt{3}, 1, 0\rangle$$

$$\mathbf{N}(1,\pi/3) = \mathbf{F}_u(1,\pi/3) \times \mathbf{F}_v(1,\pi/3) \Longrightarrow \mathbf{N}(1,\pi/3) = -2\,\mathbf{i} - 2\sqrt{3}\,\mathbf{j} + 2\,\mathbf{k}$$

and since $\mathbf{F}(2,\pi/3) = \langle 1, \sqrt{3}, 1\rangle$, an equation for the tangent plane is

$$(\langle x,y,z\rangle - \langle 1,\sqrt{3},1\rangle)\cdot\langle -2, -2\sqrt{3}, 2\rangle = 0$$

$$-2(x-1) - 2\sqrt{3}(y-\sqrt{3}) + 2(z-1) = 0 \Longleftrightarrow x + \sqrt{3}y - z = 3\ .$$

Parametric equations for the normal line are

$$x = 1-2t, \quad y = \sqrt{3} - 2\sqrt{3}t, \quad z = 1+2t\ .$$