Optimization

March 24, 2010

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How do we find points of extreme?

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<u>The converse is not true</u>: if $f_x(P_0) = f_y(P_0) = 0$, then P_0 is not necessarily a point of extreme.

Where else can one find extreme points?

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Strategy for finding extreme points:

- Check the *critical points* of f:
  - Points P_0 for which $f_x(P_0)$ or $f_y(P_0)$ does not exist;
  - Points P_0 for which $f_x(P_0) = f_y(P_0) = 0$.
- Check boundary points included in the domain.

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Find the extreme points of $f(x, y) = x^4 + y^4 - 4xy$ on $D = \mathbb{R}^2$.

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Typical mistake: $x^9 = x \iff x^8 = 1$.

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- $D(x_0, y_0) = 0$, then the test is inconclusive:
  Examples: $x^4 + y^4$, $-x^4 - y^4$, $x^4 - y^4$

Back to Example

In the example of $f(x, y) = x^4 + y^4 - 4xy$ we have

- $f_{xx} = 12x^2$;
- $f_{xy} = -4$;
- $f_{yy} = 12y^2$;
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(0,0)	0	0	-4	$-16 < 0$	Saddle point
(1,1)	12	12	-4	$144 - 16 > 0$	Local min
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But in general, global extreme points may not exist.

Extreme Value Theorem

Global extreme points are guaranteed to exist if:

- $f : D \to \mathbb{R}$ is continuous, and
- the domain D has the following properties:
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- Why does D have to be closed: to exclude $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$, $f(x, y) = (x^2 + y^2)^{-1}$. In this situation the boundary of D is $\{(0, 0)\}$ and is not included in D, so D is not closed.