Constrained Optimization

March 24, 2010

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Question: What is the maximal production that can be achieved with a budget of 16 units of money?

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$$\Longrightarrow \begin{cases} P_K(K_0, L_0) = & 3\lambda_0 \\ P_L(K_0, L_0) = & 2\lambda_0 \end{cases} \Longrightarrow \frac{P_K(K_0, L_0)}{3} = \frac{P_L(K_0, L_0)}{2}.$$

The triple (K_0, L_0, λ_0) is a solution of the system

$$\left\{ \begin{array}{ll} P_K(K,L) = & 3\lambda \\ P_L(K,L) = & 2\lambda \\ C(K,L) = & 16 \end{array} \right. \iff \left\{ \begin{array}{ll} \frac{3}{4} L^{1/4} K^{-1/4} = & 3\lambda \\ \\ \frac{1}{4} L^{-3/4} K^{3/4} = & 2\lambda \\ \\ 3K + 2L = & 16 \end{array} \right.$$

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But in this particular case:

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Geometric argument using the direction of the gradient $(\nabla P)(4,2)$.

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Consider the *Lagrange function*

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y) \ ,$$

New variable λ: *Lagrange multiplier*.

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Remarks:

- not all critical points of F give points of extreme for f;
- to decide whether a critical point gives a min, max or neither we use:
    - (harder) a special second derivative test;
    - (easier) the direction of the gradient of f.

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- Find critical points in the interior of the disk;
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Since f is differentiable everywhere, the interior extreme points are among the solutions of the system

$$\begin{cases} f_x(x, y) = & 0 \\ f_y(x, y) = & 0 \end{cases} \Longleftrightarrow \begin{cases} y = & 0 \\ x = & 0 \end{cases}$$

Find the maximum and the minimum values of $f(x, y) = xy$ on the region $D = \{(x, y) \mid |x| + |y| \leqslant 2\}$.
Extreme points on the boundary: check each of the four sides.
For the segment joining $(2, 0)$ with $(0, 2)$ we get:

$$\text{Find min/max of } f(x, y) = xy$$
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The critical points of F are the solutions of the system

$$\left\{\begin{array}{ll} F_x(x,y,\lambda) & = 0 \\ F_y(x,y,\lambda) & = 0 \\ F_\lambda(x,y,\lambda) & = 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{ll} y - \lambda & = 0 \\ x - \lambda & = 0 \\ x + y = 2 & = 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{ll} x & = 1 \\ y & = 1 \\ \lambda & = 1 \end{array}\right.$$

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Three more critical points on the boundary: $(-1, 1)$, $(-1, -1)$, $(1, -1)$. Compare the values at all points:

- the global maximum is 1, attained at $(1, 1)$ and $(-1, -1)$;
- the global minimum is -1, attained at $(1, -1)$ and $(-1, 1)$;
- the critical point $(0, 0)$ is a saddle point.

Multiple Constraints

$$\begin{aligned} \text{Find } & \min / \max f(x, y, z) \\ \text{Subject to } & g(x, y, z) = 0 \\ & h(x, y, z) = 0 \end{aligned}$$

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Condition: $(\nabla g)_P (\nabla h)_P$ are non-collinear for each intersection point P
The level surface of f through a point of extreme P_0 is tangent to the constraint curve, so $(\nabla f)(P_0)$ is perpendicular to the curve at P_0.
Constraint curve included in both surfaces $\Longrightarrow$
$(\nabla g)(P_0)$ and $(\nabla h)(P_0)$ are perpendicular to the curve $\Longrightarrow$

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Condition: $(\nabla g)_P (\nabla h)_P$ are non-collinear for each intersection point P
The level surface of f through a point of extreme P_0 is tangent to the constraint curve, so $(\nabla f)(P_0)$ is perpendicular to the curve at P_0.
Constraint curve included in both surfaces $\Longrightarrow$
$(\nabla g)(P_0)$ and $(\nabla h)(P_0)$ are perpendicular to the curve $\Longrightarrow$
there exist constants λ and μ such that

$$(\nabla f)(P_0) = \lambda (\nabla g)(P_0) + \mu (\nabla h)(P_0) \,.$$

Multiple Constraints

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The Lagrange function is in this case

$$F(x, y, z, \lambda, \mu) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$$

Example

Find the extreme points of $x + 2y$ on the intersection of the the cylinder $y^2 + z^2 = 5$ and the plane $x + y + z = 1$.

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- Constraint set is bounded and closed, function f is continuous $\Longrightarrow$ f attains its extreme on the constraint $\Longrightarrow$ $(1, -\sqrt{5/2}, \sqrt{5/2})$ corresponds to an absolute minimum and $(1, \sqrt{5/2}, -\sqrt{5/2})$ corresponds to an absolute maximum.
- The minimum value is $f(1, -\sqrt{5/2}, \sqrt{5/2}) = 1 - 2\sqrt{5/2}$ and the maximal value is $f(1, \sqrt{5/2}, -\sqrt{5/2}) = 1 + 2\sqrt{5/2}$.