Double Integrals

March 31, 2010

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$
  - states;
  - counties;
  - finer division.

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$
  - states;
  - counties;
  - finer division.

$$\text{population}(\mathcal{R}) = \sum_k \text{population}(D_k) = \sum_k \text{density}(D_k) \cdot \text{area}(D_k)$$

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$
  - states;
  - counties;
  - finer division.

$$\text{population}(\mathcal{R}) = \sum_k \text{population}(D_k) = \sum_k \text{density}(D_k) \cdot \text{area}(D_k)$$

- Density of D_k:

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$
  - states;
  - counties;
  - finer division.

$$\text{population}(\mathcal{R}) = \sum_k \text{population}(D_k) = \sum_k \text{density}(D_k) \cdot \text{area}(D_k)$$

- Density of D_k: sampling
  - $\mathcal{P} = (P_k)_k$, collection of sample points, one for each subregion;
  - density(P_k): density in a small region around P_k;
  - density$(D_k) \simeq$ density(P_k)

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$
  - states;
  - counties;
  - finer division.

$$\text{population}(\mathcal{R}) = \sum_k \text{population}(D_k) = \sum_k \text{density}(D_k) \cdot \text{area}(D_k)$$

- Density of D_k: sampling
  - $\mathcal{P} = (P_k)_k$, collection of sample points, one for each subregion;
  - $\text{density}(P_k)$: density in a small region around P_k;
  - $\text{density}(D_k) \simeq \text{density}(P_k)$

$$\text{population}(\mathcal{R}) = \sum \text{population}(D_k) \simeq \sum \text{density}(P_k)\, \text{area}(D_k) \ .$$

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$
  - states;
  - counties;
  - finer division.

$$\text{population}(\mathcal{R}) = \sum_k \text{population}(D_k) = \sum_k \text{density}(D_k) \cdot \text{area}(D_k)$$

- Density of D_k: sampling
  - $\mathcal{P} = (P_k)_k$, collection of sample points, one for each subregion;
  - $\text{density}(P_k)$: density in a small region around P_k;
  - $\text{density}(D_k) \simeq \text{density}(P_k)$

$$\text{population}(\mathcal{R}) = \sum \text{population}(D_k) \simeq \sum \text{density}(P_k)\,\text{area}(D_k)\ .$$

Cheaper census,

A Cheaper Census

To find the total population over a region $\mathcal{R}$:

- Decomposition into smaller regions, $\mathcal{D} = (D_k)_k$
  - states;
  - counties;
  - finer division.

$$\text{population}(\mathcal{R}) = \sum_k \text{population}(D_k) = \sum_k \text{density}(D_k) \cdot \text{area}(D_k)$$

- Density of D_k: sampling
  - $\mathcal{P} = (P_k)_k$, collection of sample points, one for each subregion;
  - density(P_k): density in a small region around P_k;
  - density(D_k) $\simeq$ density(P_k)

$$\text{population}(\mathcal{R}) = \sum \text{population}(D_k) \simeq \sum \text{density}(P_k)\, \text{area}(D_k)\ .$$

Cheaper census, but illegal for US Census ...

Double Integrals

Let $\mathcal{R}$ be a compact (closed, bounded) region in the plane, and let $f : \mathcal{R} \to \mathbb{R}$ be a function on $\mathcal{R}$.

Double Integrals

Let $\mathcal{R}$ be a compact (closed, bounded) region in the plane, and let $f : \mathcal{R} \to \mathbb{R}$ be a function on $\mathcal{R}$. Let $\mathcal{D} = (D_k)_k$ be a finite covering of $\mathcal{R}$

- D_k compact set, for all k;
- the boundary of D_k is a collection of smooth curves, for all k;
- two regions D_i and D_j may overlap only on their boundaries.

Double Integrals

Let $\mathcal{R}$ be a compact (closed, bounded) region in the plane, and let $f : \mathcal{R} \to \mathbb{R}$ be a function on $\mathcal{R}$. Let $\mathcal{D} = (D_k)_k$ be a finite covering of $\mathcal{R}$

- D_k compact set, for all k;
- the boundary of D_k is a collection of smooth curves, for all k;
- two regions D_i and D_j may overlap only on their boundaries.

Let $\mathcal{P} = (P_k)_k$ be a collection of sample points, with $P_k \in D_k$ for all k.

Double Integrals

Let $\mathcal{R}$ be a compact (closed, bounded) region in the plane, and let $f : \mathcal{R} \to \mathbb{R}$ be a function on $\mathcal{R}$. Let $\mathcal{D} = (D_k)_k$ be a finite covering of $\mathcal{R}$

- D_k compact set, for all k;
- the boundary of D_k is a collection of smooth curves, for all k;
- two regions D_i and D_j may overlap only on their boundaries.

Let $\mathcal{P} = (P_k)_k$ be a collection of sample points, with $P_k \in D_k$ for all k. The *Riemann sum* defined by such data is

$$\Sigma_{f,\mathcal{D},\mathcal{P}} = \sum_k f(P_k)\,\mathrm{area}(D_k)\ .$$

Double Integrals

Let $\mathcal{R}$ be a compact (closed, bounded) region in the plane, and let $f : \mathcal{R} \to \mathbb{R}$ be a function on $\mathcal{R}$. Let $\mathcal{D} = (D_k)_k$ be a finite covering of $\mathcal{R}$

- D_k compact set, for all k;
- the boundary of D_k is a collection of smooth curves, for all k;
- two regions D_i and D_j may overlap only on their boundaries.

Let $\mathcal{P} = (P_k)_k$ be a collection of sample points, with $P_k \in D_k$ for all k. The *Riemann sum* defined by such data is

$$\Sigma_{f,\mathcal{D},\mathcal{P}} = \sum_k f(P_k)\,\mathrm{area}(D_k)\,.$$

<u>Definition</u>: If the limit

$$\lim_{\mathrm{maxdiam}(\mathcal{D})\to 0} \sum_k f(P_k)\,\mathrm{area}(D_k)$$

exists and is finite, then its value is called the *double integral of f over $\mathcal{R}$ with respect to area*, and is denoted by

$$\iint_{\mathcal{R}} f(P)\,dA\,.$$

Examples

Examples

- The total population over a region $\mathcal{R}$ is:

$$\text{population}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, dA \simeq \sum_k \text{density}(P_k)\, \text{area}(D_k)\ .$$

Examples

- The total population over a region $\mathcal{R}$ is:

$$\text{population}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, dA \simeq \sum_k \text{density}(P_k)\, \text{area}(D_k)\ .$$

- Mass is the double integral of density with respect to area:

$$\text{mass}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, dA\ .$$

Examples

- The total population over a region $\mathcal{R}$ is:

$$\text{population}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, dA \simeq \sum_k \text{density}(P_k)\,\text{area}(D_k)\ .$$

- Mass is the double integral of density with respect to area:

$$\text{mass}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, dA\ .$$

- Volume under the graph of $h \colon \mathcal{R} \to [0, \infty)$

$$\text{Volume} = \iint_{\mathcal{R}} h(P)\, dA\ .$$

Examples

- The total population over a region $\mathcal{R}$ is:

$$\text{population}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, dA \simeq \sum_k \text{density}(P_k)\, \text{area}(D_k)\ .$$

- Mass is the double integral of density with respect to area:

$$\text{mass}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, dA\ .$$

- Volume under the graph of $h\colon \mathcal{R} \to [0, \infty)$

$$\text{Volume} = \iint_{\mathcal{R}} h(P)\, dA\ .$$

- Area of a region:

$$\text{Area}(\mathcal{R}) = \iint_{\mathcal{R}} 1\ dA$$

Properties

$$\iint_{\mathcal{R}} f(P)\, dA = \lim_{\operatorname{maxdiam}(\mathcal{D})\to 0} \sum_k f(P_k)\, \operatorname{area}(D_k)$$

Properties

$$\iint_{\mathcal{R}} f(P)\, dA = \lim_{\text{maxdiam}(\mathcal{D})\to 0} \sum_k f(P_k)\, \text{area}(D_k)$$

- If f is bounded and continuous, except maybe on a finite number of smooth curves, then the limit exists and is finite.

Properties

$$\iint_{\mathcal{R}} f(P)\, dA = \lim_{\text{maxdiam}(\mathcal{D})\to 0} \sum_k f(P_k)\, \text{area}(D_k)$$

- If f is bounded and continuous, except maybe on a finite number of smooth curves, then the limit exists and is finite.
- Linearity with respect to function

$$\iint_{\mathcal{R}} [\lambda f(P) + \mu g(P)]\, dA = \lambda \iint_{\mathcal{R}} f(P)\, dA + \mu \iint_{\mathcal{R}} g(P)\, dA\,.$$

Properties

$$\iint_{\mathcal{R}} f(P)\; dA = \lim_{\text{maxdiam}(\mathcal{D})\to 0} \sum_k f(P_k)\,\text{area}(D_k)$$

- If f is bounded and continuous, except maybe on a finite number of smooth curves, then the limit exists and is finite.
- Linearity with respect to function

$$\iint_{\mathcal{R}} [\lambda f(P) + \mu g(P)]\, dA = \lambda \iint_{\mathcal{R}} f(P)\, dA + \mu \iint_{\mathcal{R}} g(P)\, dA\,.$$

- Additivity with respect to domain: If $\mathcal{R}_1$ and $\mathcal{R}_2$ intersect only along the boundaries, then

$$\iint_{\mathcal{R}_1\cup\mathcal{R}_2} f(P)\, dA = \iint_{\mathcal{R}_1} f(P)\, dA + \iint_{\mathcal{R}_2} f(P)\, dA$$

Properties

$$\iint_{\mathcal{R}} f(P)\, dA = \lim_{\text{maxdiam}(\mathcal{D})\to 0} \sum_k f(P_k)\, \text{area}(D_k)$$

- If f is bounded and continuous, except maybe on a finite number of smooth curves, then the limit exists and is finite.
- Linearity with respect to function

$$\iint_{\mathcal{R}} [\lambda f(P) + \mu g(P)]\, dA = \lambda \iint_{\mathcal{R}} f(P)\, dA + \mu \iint_{\mathcal{R}} g(P)\, dA\,.$$

- Additivity with respect to domain: If $\mathcal{R}_1$ and $\mathcal{R}_2$ intersect only along the boundaries, then

$$\iint_{\mathcal{R}_1 \cup \mathcal{R}_2} f(P)\, dA = \iint_{\mathcal{R}_1} f(P)\, dA + \iint_{\mathcal{R}_2} f(P)\, dA$$

- Monotonicity property: If $m \leqslant f(P) \leqslant M$ for all P in $\mathcal{R}$, then

$$m\, \text{area}(\mathcal{R}) \leqslant \iint_{\mathcal{R}} f(P)\, dA \leqslant M\, \text{area}(\mathcal{R})\,.$$

Applications

Applications

- Average value of f on $\mathcal{R}$:

Applications

- Average value of f on $\mathcal{R}$: constant value that would produce the same accumulation.

Applications

- Average value of f on $\mathcal{R}$: constant value that would produce the same accumulation.

$$\iint_{\mathcal{R}} f(P)\,dA = \iint_{\mathcal{R}} (\text{average value})\,dA = (\text{average value})\cdot\text{area}(\mathcal{R}) \implies$$

$$\text{average value of } f \text{ on } \mathcal{R} = \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(P)\,dA\,.$$

Applications

- Average value of f on $\mathcal{R}$: constant value that would produce the same accumulation.

$$\iint_{\mathcal{R}} f(P)\,dA = \iint_{\mathcal{R}} (\text{average value})\,dA = (\text{average value})\cdot\text{area}(\mathcal{R}) \implies$$

$$\text{average value of } f \text{ on } \mathcal{R} = \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(P)\,dA\,.$$

- <u>Mean Value Theorem</u>:

  If f is continuous on $\mathcal{R}$, then there exists P_0 in $\mathcal{R}$ such that

$$f(P_0) = \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(Q)\,dA$$

Applications

- Average value of f on $\mathcal{R}$: constant value that would produce the same accumulation.

$$\iint_{\mathcal{R}} f(P)\, dA = \iint_{\mathcal{R}} (\text{average value})\, dA = (\text{average value}) \cdot \text{area}(\mathcal{R}) \implies$$

$$\text{average value of } f \text{ on } \mathcal{R} = \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(P)\, dA\,.$$

- Mean Value Theorem:

  If f is continuous on $\mathcal{R}$, then there exists P_0 in $\mathcal{R}$ such that

$$f(P_0) = \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(Q)\, dA$$

- Fundamental Theorem of Calculus (sort of ..):

  If f is continuous around P, then

$$\lim_{D \to \{P\}} \frac{1}{\text{area}(D)} \iint_{D} f(Q) dA = f(P)$$

Vectorial Integrals

$$\iint_{\mathcal{R}} \mathbf{F}(P)\, dA = \lim_{\text{maxdiam}(\mathcal{D}) \to 0} \sum_k \mathbf{F}(P_k)\, \text{area}(D_k)$$

The definition can be extended to functions with vectorial output.

<u>Example</u>: Electric force on a lamina

- Charge Q, at a fixed point
- Charge q, uniformly distributed on a planar lamina $\mathcal{R}$
- Total force on Q?

$$dq = (\text{density of charge})\, dA = \frac{q}{A(\mathcal{R})}\, dA$$

$$d\mathbf{F} = \frac{\epsilon Q dq}{|\mathbf{r}|^3}\, \mathbf{r} = \left(\epsilon \frac{Qq}{A(\mathcal{R})|\mathbf{r}|^3}\, \mathbf{r}\right)\, dA$$

$$\mathbf{F} = \iint_{\mathcal{R}} d\mathbf{F} = \iint_{\mathcal{R}} \left(\epsilon \frac{Qq}{A(\mathcal{R})|\mathbf{r}|^3}\, \mathbf{r}\right)\, dA$$

Vectorial Integrals

$$\iint_{\mathcal{R}} \mathbf{F}(P)\, dA = \lim_{\text{maxdiam}(\mathcal{D})\to 0} \sum_k \mathbf{F}(P_k)\,\text{area}(D_k)$$

The definition can be extended to functions with vectorial output.
Example: Electric force on a lamina

- Charge Q, at a fixed point
- Charge q, uniformly distributed on a planar lamina $\mathcal{R}$
- Total force on Q?

$$dq = (\text{density of charge})\, dA = \frac{q}{A(\mathcal{R})}\, dA$$

$$d\mathbf{F} = \frac{\epsilon Q dq}{|\mathbf{r}|^3}\,\mathbf{r} = \left(\epsilon \frac{Qq}{A(\mathcal{R})|\mathbf{r}|^3}\,\mathbf{r}\right)\, dA$$

$$\mathbf{F} = \iint_{\mathcal{R}} d\mathbf{F} = \iint_{\mathcal{R}} \left(\epsilon \frac{Qq}{A(\mathcal{R})|\mathbf{r}|^3}\,\mathbf{r}\right)\, dA$$

Constant Multiple Rule:

$$\iint_{\mathcal{R}} c\mathbf{F}\, dA = c \iint_{\mathcal{R}} \mathbf{F}\, dA$$

Midpoint Rule

$\mathcal{R}$: rectangle

Midpoint Rule

$\mathcal{R}$: rectangle $\Longrightarrow$ rectangular coordinate system.
$\mathcal{R} = [a, b] \times [c, d]$, element of area $dA = dx\,dy$,

$$\iint_{\mathcal{R}} f(P)\,dA = \iint_{[a,b]\times[c,d]} f(x, y)\;dx\,dy\;.$$

Midpoint Rule

$\mathcal{R}$: rectangle $\Longrightarrow$ rectangular coordinate system.
$\mathcal{R} = [a, b] \times [c, d]$, element of area $dA = dx\, dy$,

$$\iint_{\mathcal{R}} f(P)\, dA = \iint_{[a,b]\times[c,d]} f(x, y)\ dx\, dy\ .$$

If the double integral exists, then we approximate its value by a Riemann sum for a fine enough partition and a consistent choice of sample points.

Midpoint Rule

$\mathcal{R}$: rectangle $\Longrightarrow$ rectangular coordinate system.
$\mathcal{R} = [a, b] \times [c, d]$, element of area $dA = dx\,dy$,

$$\iint_{\mathcal{R}} f(P)\,dA = \iint_{[a,b]\times[c,d]} f(x, y)\;dx\,dy\ .$$

If the double integral exists, then we approximate its value by a Riemann sum for a fine enough partition and a consistent choice of sample points.

- Divide each side into n equal pieces, and obtain a covering of $\mathcal{R}$ by n^2 smaller rectangles, of sides $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{n}$.

Midpoint Rule

$\mathcal{R}$: rectangle $\Longrightarrow$ rectangular coordinate system.
$\mathcal{R} = [a, b] \times [c, d]$, element of area $dA = dx\,dy$,

$$\iint_{\mathcal{R}} f(P)\,dA = \iint_{[a,b]\times[c,d]} f(x, y)\ dx\,dy\ .$$

If the double integral exists, then we approximate its value by a Riemann sum for a fine enough partition and a consistent choice of sample points.

- Divide each side into n equal pieces, and obtain a covering of $\mathcal{R}$ by n^2 smaller rectangles, of sides $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{n}$.
- Choose the sample points as the midpoints of the smaller rectangles: take P_{ij} to be the midpoint of the rectangle D_{ij} on the i^{th} row and j^{th} column (counting from bottom left corner).

Midpoint Rule

$\mathcal{R}$: rectangle $\Longrightarrow$ rectangular coordinate system.
$\mathcal{R} = [a, b] \times [c, d]$, element of area $dA = dx\,dy$,

$$\iint_{\mathcal{R}} f(P)\,dA = \iint_{[a,b]\times[c,d]} f(x,y)\;dx\,dy\ .$$

If the double integral exists, then we approximate its value by a Riemann sum for a fine enough partition and a consistent choice of sample points.

- Divide each side into n equal pieces, and obtain a covering of $\mathcal{R}$ by n^2 smaller rectangles, of sides $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{n}$.
- Choose the sample points as the midpoints of the smaller rectangles: take P_{ij} to be the midpoint of the rectangle D_{ij} on the i^{th} row and j^{th} column (counting from bottom left corner).

$$\iint_{[a,b]\times[c,d]} f(x,y)\;dx\,dy\;\; = \lim_{n\to\infty} \sum_{1\leqslant i,j\leqslant n} f(P_{ij})\,\text{area}(D_{ij})\ ,$$

$$\iint_{[a,b]\times[c,d]} f(x,y)\;dx\,dy\;\;\simeq \sum_{1\leqslant i,j\leqslant n} f(P_{ij})\,\Delta x\,\Delta y\ .$$

Example

Use the Midpoint Rule to approximate

$$\iint_{[0,4]\times[0,2]} x^2 y \; dxdy \ ,$$

with each side divided into $n = 2$ pieces.

Example

Use the Midpoint Rule to approximate

$$\iint_{[0,4]\times[0,2]} x^2 y \; dxdy \ ,$$

with each side divided into $n = 2$ pieces.
The smaller rectangles have size $\frac{4-0}{2} \times \frac{2-0}{2} = 2 \times 1$ and area 2.

Example

Use the Midpoint Rule to approximate

$$\iint_{[0,4]\times[0,2]} x^2 y \; dxdy \; ,$$

with each side divided into $n = 2$ pieces.
The smaller rectangles have size $\frac{4-0}{2} \times \frac{2-0}{2} = 2 \times 1$ and area 2.
The midpoints are

$$P_{11} = (1, 1/2), \quad P_{12} = (1, 3/2), \quad P_{21} = (3, 1/2), \quad P_{22} = (3, 3/2) \; .$$

Example

Use the Midpoint Rule to approximate

$$\iint_{[0,4]\times[0,2]} x^2 y \; dxdy \; ,$$

with each side divided into $n = 2$ pieces.
The smaller rectangles have size $\frac{4-0}{2} \times \frac{2-0}{2} = 2 \times 1$ and area 2.
The midpoints are

$$P_{11} = (1, 1/2), \quad P_{12} = (1, 3/2), \quad P_{21} = (3, 1/2), \quad P_{22} = (3, 3/2) \; .$$

$$\iint_{[0,4]\times[0,2]} x^2 y \; dxdy \simeq$$

Example

Use the Midpoint Rule to approximate

$$\iint_{[0,4]\times[0,2]} x^2 y \; dxdy \;,$$

with each side divided into $n = 2$ pieces.
The smaller rectangles have size $\frac{4-0}{2} \times \frac{2-0}{2} = 2 \times 1$ and area 2.
The midpoints are

$$P_{11} = (1, 1/2), \quad P_{12} = (1, 3/2), \quad P_{21} = (3, 1/2), \quad P_{22} = (3, 3/2) \;.$$

$$\iint_{[0,4]\times[0,2]} x^2 y \; dxdy \simeq$$

$$\simeq f(1, 1/2) \cdot 2 + f(3, 1/2) \cdot 2 + f(1, 3/2) \cdot 2 + f(3, 3/2) \cdot 2 =$$

$$= \left(1 \cdot \frac{1}{2} \cdot 2 + 9 \cdot \frac{1}{2} \cdot 2\right) + \left(1 \cdot \frac{3}{2} \cdot 2 + 9 \cdot \frac{3}{2} \cdot 2\right) =$$

$$= 1 + 9 + 3 + 27 = 40 \;.$$

Iterated Integrals

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy \simeq \sum_{1\leqslant i,j\leqslant n} f(P_{ij})\,\Delta x\,\Delta y =$$

$$= \sum_{1\leqslant i,j\leqslant n} f(x_i,y_j)\,\Delta x\,\Delta y = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\,\Delta y\ .$$

Iterated Integrals

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy \simeq \sum_{1\leqslant i,j\leqslant n} f(P_{ij})\,\Delta x\,\Delta y =$$

$$= \sum_{1\leqslant i,j\leqslant n} f(x_i,y_j)\,\Delta x\,\Delta y = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\,\Delta y\ .$$

The sum with index i is a Riemann sum for computing the integral

$$g(y_j) = \int_{x=a}^{x=b} f(x,y_j)\,dx$$

Iterated Integrals

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy \simeq \sum_{1\leqslant i,j\leqslant n} f(P_{ij})\,\Delta x\,\Delta y =$$

$$= \sum_{1\leqslant i,j\leqslant n} f(x_i,y_j)\,\Delta x\,\Delta y = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\,\Delta y\;.$$

The sum with index i is a Riemann sum for computing the integral

$$g(y_j) = \int_{x=a}^{x=b} f(x,y_j)\,dx$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\Delta y \simeq \sum_{j=1}^{n} g(y_j)\Delta y \simeq \int_{y=c}^{y=d} g(y)\,dy$$

Iterated Integrals

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy \simeq \sum_{1\leqslant i,j\leqslant n} f(P_{ij})\,\Delta x\,\Delta y =$$

$$= \sum_{1\leqslant i,j\leqslant n} f(x_i,y_j)\,\Delta x\,\Delta y = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\,\Delta y\ .$$

The sum with index i is a Riemann sum for computing the integral

$$g(y_j) = \int_{x=a}^{x=b} f(x,y_j)\,dx$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\,\Delta y \simeq \sum_{j=1}^{n} g(y_j)\Delta y \simeq \int_{y=c}^{y=d} g(y)\,dy$$

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_{y=c}^{y=d} g(y)\,dy = \int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)\,dy$$

Fubini's Theorem

Let $f\colon [a,b]\times[c,d]\to\mathbb{R}$ be a function that is

- bounded
- continuous, except maybe on a finite number of smooth curves.

If the iterated integrals exist, then

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dxdy = \int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)\,dy =$$
$$= \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)\,dx\,.$$

The iterated integrals exists if f is continuous.

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy$$

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\, dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.
- For a fixed value $y = y_0$, x takes values

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\, dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.
- For a fixed value $y = y_0$, x takes values between $a = 1$ and $b = 2$.

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.
- For a fixed value $y = y_0$, x takes values between $a = 1$ and $b = 2$.

Then the double integral can be set up as the following iterated integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy = \int_{y=2}^{y=3} \left(\int_{x=1}^{x=2} (2x + 3y^2)\ dx \right)\ dy$$

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.
- For a fixed value $y = y_0$, x takes values between $a = 1$ and $b = 2$.

Then the double integral can be set up as the following iterated integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy = \int_{y=2}^{y=3} \left(\int_{x=1}^{x=2} (2x + 3y^2)\ dx \right)\ dy$$

Integral with respect to x

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.
- For a fixed value $y = y_0$, x takes values between $a = 1$ and $b = 2$.

Then the double integral can be set up as the following iterated integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\ dxdy = \int_{y=2}^{y=3} \left(\int_{x=1}^{x=2} (2x + 3y^2)\ dx \right)\ dy$$

Integral with respect to $x \implies y$ is a constant:

$$g(y) = \int_{x=1}^{x=2} (2x+3y^2)\ dx = (x^2 + 3y^2x)\Big|_{x=1}^{x=2} = (4+6y^2)-(1+3y^2) = 3+3y^2\ .$$

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\, dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.
- For a fixed value $y = y_0$, x takes values between $a = 1$ and $b = 2$.

Then the double integral can be set up as the following iterated integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\, dxdy = \int_{y=2}^{y=3} \left(\int_{x=1}^{x=2} (2x + 3y^2)\, dx \right)\, dy$$

Integral with respect to $x \implies y$ is a constant:

$$g(y) = \int_{x=1}^{x=2} (2x+3y^2)\, dx = (x^2 + 3y^2 x)\Big|_{x=1}^{x=2} = (4+6y^2)-(1+3y^2) = 3+3y^2\ .$$

Continuing the computation we get

Example

Use iterated integrals to compute the following integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\; dxdy$$

- For (x, y) in $[1, 2] \times [2, 3]$, y takes values between $c = 2$ and $d = 3$.
- For a fixed value $y = y_0$, x takes values between $a = 1$ and $b = 2$.

Then the double integral can be set up as the following iterated integral:

$$\iint_{[1,2]\times[2,3]} (2x + 3y^2)\; dxdy = \int_{y=2}^{y=3} \left(\int_{x=1}^{x=2} (2x + 3y^2)\; dx \right)\; dy$$

Integral with respect to $x \implies y$ is a constant:

$$g(y) = \int_{x=1}^{x=2} (2x+3y^2)\; dx = (x^2 + 3y^2 x)\Big|_{x=1}^{x=2} = (4+6y^2)-(1+3y^2) = 3+3y^2\ .$$

Continuing the computation we get

$$\iint_{[1,2]\times[2,3]} (2x+3y^2)\; dxdy = \int_{y=2}^{y=3} (3+3y^2)\; dy = (3y + y^3)\Big|_{y=2}^{y=3} = 36-14 = 22\ .$$

More General Regions

What makes the iterated integral method work over rectangular regions?

More General Regions

What makes the iterated integral method work over rectangular regions?

> *For all values of one variable, all slices with respect to that variable are intervals in the other variable.*

More General Regions

What makes the iterated integral method work over rectangular regions?

For all values of one variable, all slices with respect to that variable are intervals in the other variable.

If the variable is y, then:

- we can integrate over each slice with respect to x,
- obtaining a function that depends only on the location of the slice,
- given by the $y-$value,
- then we integrate the result again with respect to y.

More General Regions

What makes the iterated integral method work over rectangular regions?

> *For all values of one variable, all slices with respect to that variable are intervals in the other variable.*

If the variable is y, then:

- we can integrate over each slice with respect to x,
- obtaining a function that depends only on the location of the slice,
- given by the $y-$value,
- then we integrate the result again with respect to y.

We can apply the same procedure if the slices are intervals, with endpoints depending continuously on the location of the slice.

More General Regions

What makes the iterated integral method work over rectangular regions?

> *For all values of one variable, all slices with respect to that variable are intervals in the other variable.*

If the variable is y, then:

- we can integrate over each slice with respect to x,
- obtaining a function that depends only on the location of the slice,
- given by the $y-$value,
- then we integrate the result again with respect to y.

We can apply the same procedure if the slices are intervals, with endpoints depending continuously on the location of the slice.

- Regions of type I: vertical slices are segments.
- Regions of type II: horizontal slices are segments.

Strategy: Type I Regions

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.
- Draw a generic vertical slice at some value x between a and b.

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.
- Draw a generic vertical slice at some value x between a and b.
- Find the lowest point on that slice, $(x, g_1(x))$ and the highest point, $(x, g_2(x))$.

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.
- Draw a generic vertical slice at some value x between a and b.
- Find the lowest point on that slice, $(x, g_1(x))$ and the highest point, $(x, g_2(x))$.

The region is the region bounded by:

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.
- Draw a generic vertical slice at some value x between a and b.
- Find the lowest point on that slice, $(x, g_1(x))$ and the highest point, $(x, g_2(x))$.

The region is the region bounded by:

- vertical lines $x = a$ and $x = b$;

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.
- Draw a generic vertical slice at some value x between a and b.
- Find the lowest point on that slice, $(x, g_1(x))$ and the highest point, $(x, g_2(x))$.

The region is the region bounded by:

- vertical lines $x = a$ and $x = b$;
- the graphs of $y = g_1(x)$ and $y = g_2(x)$, with $g_1, g_2 \colon [a, b] \to \mathbb{R}$:

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.
- Draw a generic vertical slice at some value x between a and b.
- Find the lowest point on that slice, $(x, g_1(x))$ and the highest point, $(x, g_2(x))$.

The region is the region bounded by:

- vertical lines $x = a$ and $x = b$;
- the graphs of $y = g_1(x)$ and $y = g_2(x)$, with $g_1, g_2 \colon [a, b] \to \mathbb{R}$:

$$\mathcal{R} = \{(x, y) \mid a \leqslant x \leqslant b \, , \; g_1(x) \leqslant y \leqslant g_2(x)\} \, .$$

Strategy: Type I Regions

- Identify the leftmost point, $(a, *)$, and the rightmost point, $(b, *)$.
- Draw a generic vertical slice at some value x between a and b.
- Find the lowest point on that slice, $(x, g_1(x))$ and the highest point, $(x, g_2(x))$.

The region is the region bounded by:

- vertical lines $x = a$ and $x = b$;
- the graphs of $y = g_1(x)$ and $y = g_2(x)$, with $g_1, g_2 \colon [a, b] \to \mathbb{R}$:

$$\mathcal{R} = \{(x, y) \mid a \leqslant x \leqslant b\ ,\ g_1(x) \leqslant y \leqslant g_2(x)\}\ .$$

$$\iint_{\mathcal{R}} f(x, y)\, dxdy = \int_{x=a}^{x=b} \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy \right) dx$$

Strategy: Type II Regions

- Identify the lowest point, $(*, c)$, and the highest point, $(*, d)$.
- Draw a generic horizontal slice at some value y between c and d.
- Find the leftmost point on that slice, $(h_1(y), y)$ and the rightmost point, $(h_2(y), y)$.

The region is bounded by:

- horizontal lines $y = c$ and $y = d$
- graphs of $x = h_1(y)$ and $x = h_2(y)$, with $h_1, h_2 \colon [c, d] \to \mathbb{R}$:

$$\mathcal{R} = \{(x, y) \mid c \leqslant y \leqslant d \ , \ h_1(y) \leqslant x \leqslant h_2(y)\} \ .$$

$$\iint_{\mathcal{R}} f(x, y)\, dxdy = \int_{y=c}^{y=d} \left(\int_{x=h_1(y)}^{x=h_2(y)} f(x, y)\, dx \right) dy$$

Examples

- $\mathcal{R}_1$: region bounded by $y = 2x$ and $y = x^2$. Compute

$$\iint_{\mathcal{R}_1} (x^2 + y^2)\, dxdy$$

- $\mathcal{R}_2$: region bounded by $y = x - 1$ and $y^2 = 2x + 6$. Compute

$$\iint_{\mathcal{R}_2} xy\, dxdy$$

- $\mathcal{R}$: region bounded by $y = (x+1)^2$, $x = y - y^3$, the line $x = -1$ and the line $y = -1$. Set-up iterated integrals for

$$\iint_{\mathcal{R}} f(x,y)\, dA\,.$$

- How do we compute

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\; dy \qquad , \qquad \iint_{[0,\infty)\times[0,\infty)} e^{-x-y}\, dxdy$$