Double Integrals in Polar Coordinates

April 5, 2010

Moment of Inertia (w.r.t. an axis)

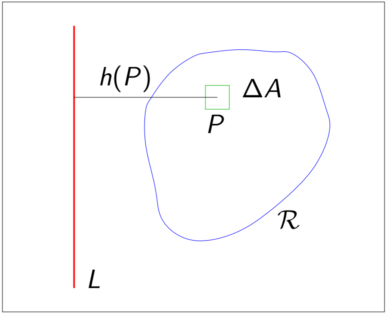
$$I_L = \text{mass} \cdot \text{distance}^2 = Md^2$$

For a lamina (thin)

- occupying a region $\mathcal{R}$,
- variable density $\rho \colon \mathcal{R} \to (0, \infty)$,
- distance to axis $h \colon \mathcal{R} \to [0, \infty)$, $h(P) = \text{distance}(P, L)$

$$dI_L = h^2\, dm = h^2\, \rho\, dA$$

$$I_L = \iint_{\mathcal{R}} dI_L = \iint_{\mathcal{R}} h^2\, dm = \iint_{\mathcal{R}} h^2\, \rho\, dA$$



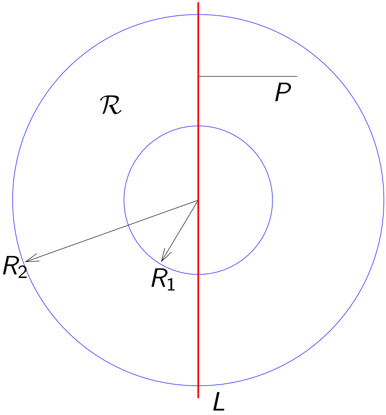
Example

Moment of inertia of a ring-like lamina with

- inner radius R_1
- outer radius R_2
- and constant density ρ

about an axis that passes through its center:

$$I_L = \iint_{\mathcal{R}} \rho\, h^2(P) dA \,.$$



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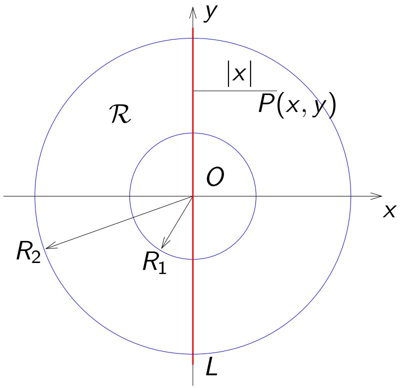
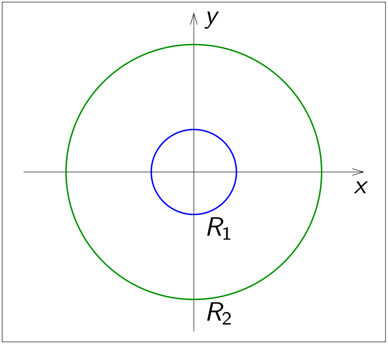
In rectangular coordinates:

$$I_L = \iint_{\mathcal{R}} \rho x^2\ dx\, dy$$



Computation in polar coordinates



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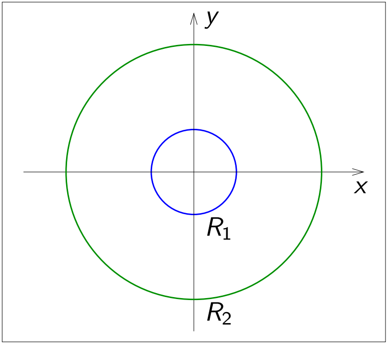

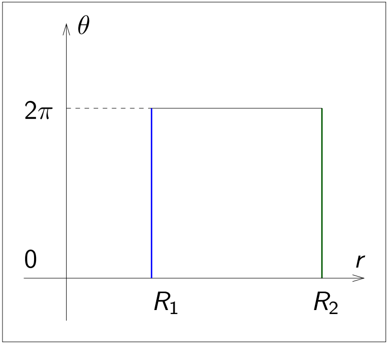

$$\mathcal{R}_{polar} = [R_1, R_2] \times [0, \theta] = \{(r, \theta) \mid R_1 \leqslant r \leqslant R_2, 0 \leqslant \theta \leqslant 2\pi\}$$

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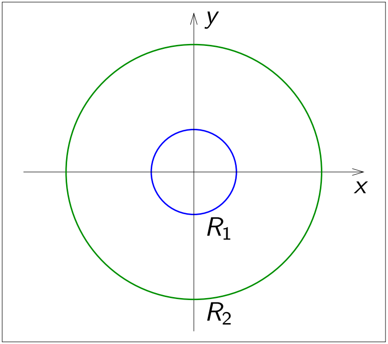
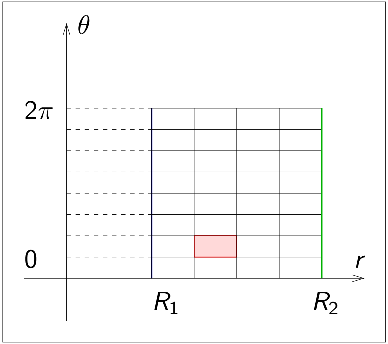
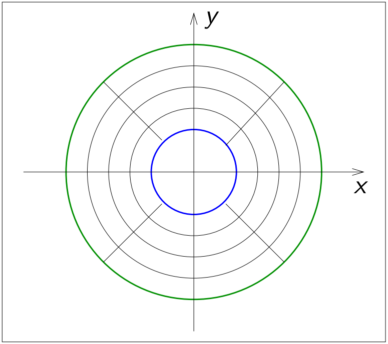


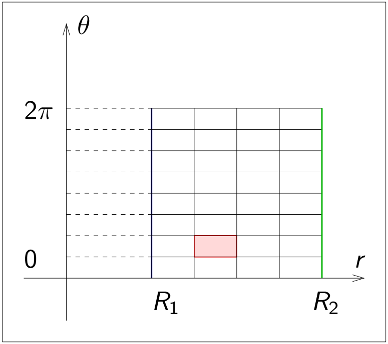


$$T : \mathcal{R}_{polar} \to \mathcal{R}_{rectangular}$$

We divide the rectangular polar region into smaller rectangular polar regions, $D_1^p, \ldots, D_N^p$.

Computation in polar coordinates





$$D_k = T(D_k^p)$$

The corresponding subregions of $\mathcal{R}$, for $k = 1, \ldots, N$ are regions that look like sectors of rings.

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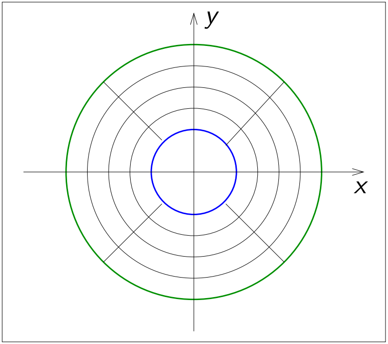

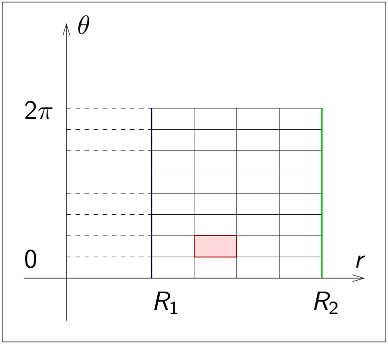

$$P_k = T(r_k, \theta_k)$$

For each subregion D_k of $\mathcal{R}$, pick a sample point P_k in D_k, and let (r_k, θ_k) be the corresponding point in the polar subregion D_k^p.

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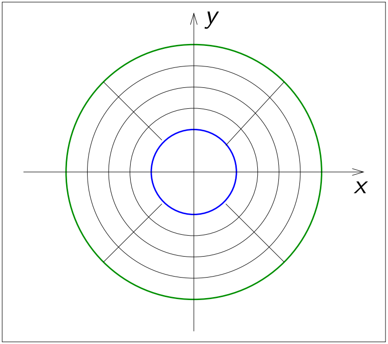
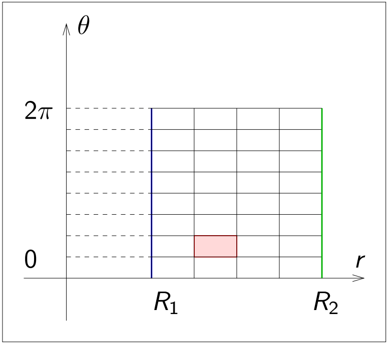
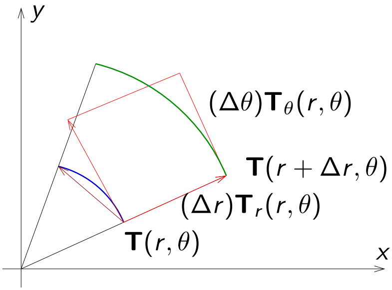




$$\iint_{\mathcal{R}} f(P)dA \simeq \sum_{k=1}^{N} f(P_k)\cdot \text{area}(D_k) = \sum_{k=1}^{N} f(T(r_k,\theta_k))\cdot \text{area}(T(D_k^p))\ .$$

Area Scaling



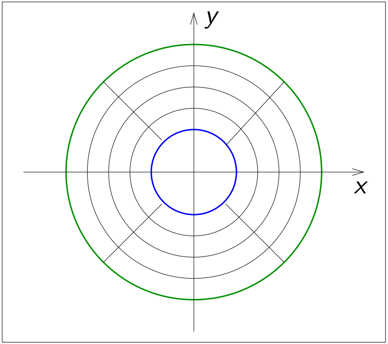
$$\langle x, y\rangle = \mathbf{T}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$$

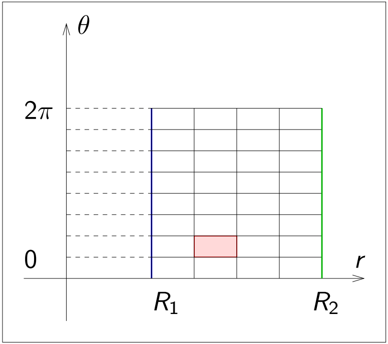
$$\mathbf{T}(r+\Delta r,\theta) - \mathbf{T}(r,\theta) \simeq (\Delta r)\mathbf{T}_r(r,\theta)$$

$$\mathbf{T}(r,\theta+\Delta\theta) - \mathbf{T}(r,\theta) \simeq (\Delta\theta)\mathbf{T}_\theta(r,\theta)$$

$$\text{Area}(D) \simeq |\mathbf{T}_r \times \mathbf{T}_\theta|\Delta r\Delta\theta = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \Delta r\,\Delta\theta = r\Delta r\Delta\theta\ .$$

Change of Variables





$$\iint_{\mathcal{R}} f(P)dA \simeq \sum_{k=1}^{N} f(P_k) \cdot \text{area}(D_k) = \sum_{k=1}^{N} f(T(r_k, \theta_k)) \cdot r_k \Delta r \Delta \theta \ .$$

$$\iint_{\mathcal{R}} f(P)dA = \iint_{\mathcal{R}_{polar}} f(r\cos\theta, r\sin\theta) \ \cdot \ r\, dr\, d\theta \ .$$

Example, continued

$$\iint_{\mathcal{R}} f(P)dA = \iint_{\mathcal{R}_{polar}} f(r\cos\theta, r\sin\theta) \;\cdot\; r\,dr\,d\theta\ .$$

Returning to the computation of the moment of inertia we get

$$I_L = \iint_{\mathcal{R}} f(P)dA = \iint_{\mathcal{R}} x^2\rho\,dxdy =$$

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$$\begin{aligned}
I_L &= \iint_{\mathcal{R}} f(P)dA = \iint_{\mathcal{R}} x^2\rho\,dxdy = \iint_{\mathcal{R}_{polar}} (r\cos\theta)^2 \cdot r\,dr\,d\theta = \\
&= \iint_{[R_1,R_2]\times[0,2\pi]} \rho r^3\cos^2\theta\,dr\,d\theta = \\
&= \rho\left(\int_{r=R_1}^{r=R_2} r^3\,dr\right)\left(\int_{\theta=0}^{\theta=2\pi}\frac{1+\cos 2\theta}{2}\,d\theta\right) = \\
&= \frac{\pi\rho(R_2^4-R_1^4)}{4} = \rho\pi(R_2^2-R_1^2)\cdot\frac{R_1^2+R_2^2}{4} = \frac{m(R_1^2+R_2^2)}{4}\ .
\end{aligned}$$

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Then

$$\iint_{\mathcal{R}} f(P)\, dA = \iint_{\mathcal{R}_{polar}} g(r,\theta) \cdot r\, dr\, d\theta \,.$$

Change of Coordinates

In many cases, the region $\mathcal{R}$ is given in rectangular coordinates, $\mathcal{R}_{rectangular}$ and the integral is

$$\iint_{\mathcal{R}_{rectangular}} f(x, y)\, dxdy \ .$$

To change to polar coordinates:

$$\iint_{\mathcal{R}_{rectangular}} f(x, y)\, dxdy \rightsquigarrow \iint_{\mathcal{R}} f(P)\, dA \rightsquigarrow \iint_{\mathcal{R}_{polar}} g(r, \theta)\, r\, drd\theta \ .$$

Strategy:

- Change region $\mathcal{R}_{rectangular} \rightarrow \mathcal{R} \rightarrow \mathcal{R}_{polar}$;
- Change function $f(x, y) \rightarrow f(r\cos\theta, r\sin\theta) = g(r, \theta)$;
- Change element of area: $dxdy \rightarrow r\, drd\theta$.

When Do We Use Polar Coordinates?

$$\iint_{\mathcal{R}_{rectangular}} f(x,y)\,dxdy \rightsquigarrow \iint_{\mathcal{R}} f(P)\,dA \rightsquigarrow \iint_{\mathcal{R}_{polar}} g(r,\theta)\,r\,drd\theta\ .$$

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- $r\,g(r,\theta) = r\,f(r\cos\theta, r\sin\theta)$ is not too complicated.

Example

Let $\mathcal{R} = \mathcal{R}_{rectangular}$ be the region left of $y-$axis and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Compute

$$\iint_{\mathcal{R}} (x + y)\, dxdy \ .$$

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Then the integral is

$$\iint_{\mathcal{R}} (x + y)\, dxdy = \iint_{[1,2]\times[\pi/2, 3\pi/2]} (r\cos\theta + r\sin\theta) \cdot rdr\, d\theta =$$

$$= \left(\int_1^2 r^2\, dr \right) \left(\int_{\pi/2}^{3\pi/2} (\sin\theta + \cos\theta)\, d\theta \right) = -\frac{14}{3}\ .$$

Improper Integrals in Polar Coordinates

$\mathcal{Q}$: first quadrant, $[0, \infty) \times [0, \infty)$. Compute

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$$= \left(\int_{\theta=0}^{\theta=\pi/2} d\theta\right)\left(\int_{r=0}^{r\to\infty} re^{-r^2}\,dr\right) = \frac{\pi}{2}\;\frac{-e^{-r^2}}{2}\Bigg|_{r=0}^{r\to\infty} = \frac{\pi}{4}\,.$$

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$$= \left(\int_{\theta=0}^{\theta=\pi/2} d\theta\right)\left(\int_{r=0}^{r\to\infty} re^{-r^2}\,dr\right) = \frac{\pi}{2}\;\frac{-e^{-r^2}}{2}\Bigg|_{r=0}^{r\to\infty} = \frac{\pi}{4}\,.$$

Application:

$$\frac{\pi}{4} = \iint_{[0,\infty)\times[0,\infty)} e^{-x^2}\,e^{-y^2}\,dxdy = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx =$$

Improper Integrals in Polar Coordinates

$\mathcal{Q}$: first quadrant, $[0,\infty)\times[0,\infty)$. Compute

$$\iint_{Q} e^{-x^2-y^2}\, dxdy$$

$$\iint_{\mathcal{Q}} e^{-x^2-y^2}\, dx\, dy = \iint_{[0,\infty)\times[0,\pi/2]} e^{-r^2}\, r\, dr\, d\theta =$$

$$= \left(\int_{\theta=0}^{\theta=\pi/2} d\theta\right)\left(\int_{r=0}^{r\to\infty} re^{-r^2}\, dr\right) = \frac{\pi}{2}\; \frac{-e^{-r^2}}{2}\Bigg|_{r=0}^{r\to\infty} = \frac{\pi}{4}\,.$$

<u>Application</u>:

$$\frac{\pi}{4} = \iint_{[0,\infty)\times[0,\infty)} e^{-x^2}\, e^{-y^2}\, dxdy = \left(\int_0^{\infty} e^{-x^2} dx\right)\left(\int_0^{\infty} e^{-y^2} dy\right)$$

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-x^2}\, dx = 2\left(\iint_{\mathcal{Q}_1} e^{-x^2-y^2}\, dx\, dy\right)^{1/2} = \sqrt{\pi}\,.$$