Line Integrals

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Integrals

We have so far studied accumulations (integrals) over:

- intervals on a line;
- planar regions in the plane;
- solid regions in space.

Regions that have the same dimension as their ambient space.

What if the region has a lower dimension then the ambient space?

- curve (1D region) embedded in a plane (2D) or in space (3D)
- surface (2D region) embedded in space (3D).

Example

Accumulations along a curve with respect to arclength

- C be a continuous curve, with both endpoints included;
- C has a piecewise smooth regular parametrization $\mathbf{r}\colon [a,b] \to C$;
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  - the mass m of the wire: accumulation of dm.

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Given:

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- The total accumulation is approximated by the Riemann sum

$$\sum_{k=1}^{N} f(P_k) \cdot \text{length}(D_k)$$

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The line integral always exists if

- f is a continuous functions, or
- f is bounded and continuous except at a finite number of points.

Parametrizations and Computations

$\mathbf{r}\colon [a,b] \to C$: regular, piecewise smooth parametrization of C

$$ds = |\mathbf{r}'(t)| dt$$

$$\int_C f(P)\, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)|\, dt\ .$$

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In rectangular coordinates:

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^2 \quad , \quad \mathbf{r}(t) = \langle x(t), y(t) \rangle$$

$$ds = |\mathbf{r}'(t)|dt = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$\int_C f(P)\, ds = \int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt\ .$$

Example

C: first quadrant part of the circle of radius 2 centered at origin.

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$$f(x(t), y(t)) = (x(t))^2 y(t) = 8\cos^2 t \sin t \; .$$

$$\int_C x^2 y \, ds = \int_{t=0}^{t=\pi/2} f(x(t), y(t)) |\mathbf{r}'(t)| \, dt = \int_{t=0}^{t=\pi/2} 8\cos^2 t \sin t \; 2dt =$$

$$= 16 \, \frac{-\cos^3 t}{3} \Bigg|_{t=0}^{t=\pi/2} = \frac{16}{3} \; .$$

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- Line integral of $\mathbf{F}$ along C (in all dimensions):

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \,, \quad \text{with} \quad \mathbf{dr} = \mathbf{T}\, ds \,.$$

Work done by a force field $\mathbf{F}$: $dW = \mathbf{F} \cdot \mathbf{dr} = \mathbf{F} \cdot \mathbf{T}\, ds$

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  Flux across a membrane: $\mathbf{F} \cdot \mathbf{N}$ is the normal component of $\mathbf{F}$.

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$$\mathbf{F} \cdot \mathbf{dr} = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right)\; dt\ .$$

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_{t=a}^{t=b} \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right)\; dt$$

Example

Work done by $\mathbf{F} = \langle x, -y \rangle = x\,\mathbf{i} - y\,\mathbf{j}$ in moving a particle from $(1, 0)$ to $(0, 1)$ along the quarter of the unit circle contained in the first quadrant.

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A parametrization of C compatible with the given orientation is

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What if the parametrization is *not* compatible with the orientation?

1-Forms

We return to $\mathbf{F} \cdot \mathbf{dr}$. Since $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, we have $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j}$.
If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$, then

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An expression of the type

$$\begin{aligned} \omega &= P(x)\,dx && \text{(in 1D)} \\ \omega &= P(x, y)\,dx + Q(x, y)\,dy && \text{(in 2D)} \\ \omega &= P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz && \text{(in 3D)} \end{aligned}$$

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  on the segment with endpoints a *and* b
  oriented from a *to* b.

Integrals of 1-Forms

- $\omega = P(x, y)\, dx + Q(x, y)\, dy$ is a 1-form;
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Re-parametrization of the interval $\simeq$ Substitution rule for integrals

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Compute

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1-Forms in Polar Coordinates

In polar coordinates:

$$x = r\cos\theta \Longrightarrow dx = \cos\theta dr - r\sin\theta d\theta$$

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Similarly:

$$\frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = \frac{1}{r}\,dr = d(\ln r)$$

$$\oint_C \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = \oint_C d(\ln r) = 0$$