Conservative Fields

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Inverse Square Distance Fields

- O: fixed point in space
- $\mathbf{F}$: vector field (defined away from O) given by

$$\mathbf{F}(A) = \frac{1}{|OA|^2}\widehat{\mathbf{AO}} = -\frac{1}{|OA|^3}\,\mathbf{OA}\ .$$

- C be a smooth curve with endpoints A and B.
- Work done by the field $\mathbf{F}$ in moving a particle from A to B along C

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$\mathbf{r}\colon [a,b] \to C$: parametrization of C, with $A = \mathbf{r}(a)$ and $B = \mathbf{r}(b)$

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Change of variable $u = |\mathbf{r}(t)|^2$, $du = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)\, dt$

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The line integral:

- depends only on the endpoints A and B;
- does not depend on the path C from A to B.

Conservative Fields

Definition: A vector field $\mathbf{F}$ is called *conservative* if for every pair of points A and B and every pair of paths C_1 and C_2 joining A to B we have

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Equivalence: $C = C_1 \cup (-C_2)$ starts and ends at A and

$$\int_{C} \mathbf{F} \cdot \mathbf{dr} = \int_{C_1} \mathbf{F} \cdot \mathbf{dr} - \int_{C_2} \mathbf{F} \cdot \mathbf{dr} \ .$$

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$$\int_C (\nabla f) \cdot \mathbf{dr} = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t))\, dt = f(B) - f(A) \ ,$$

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  depends only on A and B, but not on $C \Longrightarrow \mathbf{F}$ is conservative.

A Criterion in Rectangular Coordinates

$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$: smooth field
If $\mathbf{F}$ is a gradient field, then $\mathbf{F} = \nabla f$, hence

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If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a gradient field, then

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If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a gradient field, then

$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y\ .$$

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Unfortunately, NO!

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}\ ,$$

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Definition: A domain D is called *simply connected* if every closed loop in D can be deformed ("lassoed") into a point inside D.

If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ is defined on a simply connected domain D and $P_y(x,y) = Q_x(x,y)$ over D, then $\mathbf{F}$ is a gradient field.
Similar for $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

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C: any smooth curve joining the points $(1,0)$ and $(0,1)$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (\nabla f)\cdot d\mathbf{r} = f(1,0) - f(0,1) = (3+a) - (-1+a) = 4\ .$$

Exact 1-Forms

Let $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ be a vector field, and $\omega = \mathbf{F} \cdot \mathbf{dr} = P(x, y)dx + Q(x, y)dy$ the corresponding 1-form.

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Compare to: $df = f'(x)\,dx$

$$\int_a^b df = \int_a^b f'(x)\,dx = f(b) - f(a)\ .$$

Conservation of Energy

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How do you reconcile the two formulas?