Green's Theorem

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There are two fundamental effects:

1. Normal component of **F** $\Longrightarrow$ moves matter across the boundary.
   Accumulation: *flux across* C;
2. Tangential component of **F** $\Longrightarrow$ moves matter along the boundary.
   Accumulation: *circulation along* C.

Flux and Divergence

- Convention: we measure the *outward* flux;
- $\mathbf{N}$: unit vector, normal to C and pointing outwards from D;

$$\text{Outward Flux } = \oint_C \mathbf{F} \cdot \mathbf{N} ds = \oint_C \mathbf{F} \cdot \mathbf{dn} \, ,$$

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Divergence of $\mathbf{F}$ at p: the density of flux

$$(\operatorname{div} \mathbf{F})(p) = \lim_{D \to \{p\}} \frac{\text{Outward flux across boundary}}{\text{Area}(D)} =$$
$$= \lim_{D \to \{p\}} \frac{1}{\text{Area}(D)} \oint_C \mathbf{F} \cdot \mathbf{dn}$$

Circulation and Flatland Curl

- Convention: the plane has a predetermined orientation.
- a regular parametrization of C is *positive* if C is followed in a counterclockwise direction $\Longleftrightarrow \{\mathbf{N}, \mathbf{T}\}$ is positively oriented

$$\text{Circulation } = \oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C \mathbf{F} \cdot \mathbf{dr} \ ,$$

where $\mathbf{dr} = \mathbf{r}'(t)dt = \mathbf{T}ds$ is the tangent vectorial element of arclength.

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The *Flatland curl* of $\mathbf{F}$ at p: the density of circulation

$$(\mathbf{curl}_{\mathbf{k}} \mathbf{F})(p) = \lim_{D \to \{p\}} \frac{\text{Circulation along boundary}}{\text{Area}(D)} =$$
$$= \lim_{D \to \{p\}} \frac{1}{\text{Area}(D)} \oint_C \mathbf{F} \cdot d\mathbf{r} \,.$$

Computations in Rectangular Coordinates

- $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$
- $\mathbf{r}\colon [a, b] \to \mathbb{R}^2$ be a positive, piecewise smooth parametrization by arclength of the boundary $C = \partial D$

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}\ .$$

$$\mathbf{T}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} \qquad \text{and} \qquad \mathbf{N}(t) = y'(t)\mathbf{i} - x'(t)\mathbf{j}$$

Then

$$\oint_C \mathbf{F} \cdot \mathbf{dn} = \int_a^b \left[P(x, y)y' - Q(x, y)x'\right]\, dt = \oint_C P(x, y)dy - Q(x, y)dx$$

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Key fact: If f is a continuous function defined around p, then

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\lim_{D \to \{p\}} \frac{1}{\text{Area}(D)} \iint_D f(q)dA = f(p) \ .
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We need to convert the line integrals into double integrals.

Green's Theorem

Theorem (Green)

Let D be a region whose boundary C is piecewise smooth and positively oriented. If P and Q have continuous partial derivatives in an open region around D, then

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_D \left[\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right] dxdy \; .$$

Companion formula:

$$\oint_C P(x,y)dy - Q(x,y)dx = \iint_D \left[\frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y)\right] dxdy \; .$$

Divergence and Flatland Curl

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Coordinate-free formulations of Green's theorem:

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\oint_{\partial D} \mathbf{F}\cdot \mathbf{dr} = \iint_D \mathbf{curl_k}\,\mathbf{F}\,dA \qquad \text{and} \qquad \oint_{\partial D} \mathbf{F}\cdot \mathbf{dn} = \iint_D \mathrm{div}\,\mathbf{F}\,dA
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Interpretation of Divergence

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{dn} = \iint_D \operatorname{div} \mathbf{F} \, dA$$

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  - the amount pushed inside in some places is equal to the amount pushed outside somewhere else;
  - The field $\mathbf{F}$ is *incompressible* or *solenoidal*.

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On simply connected regions:

$$\text{Irrotational field} \Longleftrightarrow \text{Conservative field}$$

Divergence and Curl in Polar Coordinates

Suppose that in polar coordinates (r, θ), the field $\mathbf{F}$ is given by

$$\mathbf{F}(r, \theta) = P(r, \theta)\mathbf{e}_r + Q(r, \theta)\mathbf{e}_\theta \ ,$$

where $\mathbf{e}_r = \mathbf{e}_r(r, \theta)$ and $\mathbf{e}_\theta = \mathbf{e}_\theta(r, \theta)$ are the unit polar coordinate vectors.

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- If $\mathbf{F}$ is a radial field given in polar coordinates by $\mathbf{F} = f(r, \theta)\mathbf{e}_r$, then

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- Radial fields $\mathbf{F}(r,\theta) = f(r)\mathbf{e}_r$ are irrotational.

Examples

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- In rectangular coordinates centered at p:

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$$\mathbf{curl_k}\,\mathbf{F} = \frac{\partial(\omega x)}{\partial x} - \frac{\partial(-\omega y)}{\partial y} = 2\omega\ ;$$

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The divergence of this field is identically zero, so the field is incompressible.

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$$\mathbf{F}(r,\theta) = ar^{-2}\mathbf{e}_r$$

$$\operatorname{div}\mathbf{F} = \frac{1}{r}\frac{\partial(rar^{-2})}{\partial r} + \frac{1}{r}\frac{\partial 0}{\partial \theta} = -\frac{a}{r^3}\,.$$

$$\mathbf{curl_k}\,\mathbf{F} = 0\,.$$