Surface Integrals

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- How do we compute surface integrals?

Surface Area

- $\varphi\colon D \to \mathbb{R}^3$ be a local parametrization;
- (u_0, v_0) a point in the parameter space and $P = \varphi(u_0, v_0)$;
- $B = [u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$: small rectangle in the parameter space;
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$E \simeq$ curvilinear box on S with one vertex at $\boldsymbol{\varphi}(u_0, v_0)$ and directions

$$\boldsymbol{\varphi}(u_0 + \Delta u, v_0) - \boldsymbol{\varphi}(u_0, v_0) \simeq \boldsymbol{\varphi}_u(u_0, v_0)\Delta u$$
$$\boldsymbol{\varphi}(u_0, v_0 + \Delta v) - \boldsymbol{\varphi}(u_0, v_0) \simeq \boldsymbol{\varphi}_v(u_0, v_0)\Delta v$$

$$\text{area}(E) \simeq |\boldsymbol{\varphi}_u(u_0, v_0)\Delta u \times \boldsymbol{\varphi}_v(u_0, v_0)\Delta v| = |\boldsymbol{\varphi}_u(u_0, v_0) \times \boldsymbol{\varphi}_v(u_0, v_0)|\Delta u \Delta v\,.$$

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If B is a compact (bounded and closed) subset of D, then $S = \varphi(B)$ is a compact surface in space and

$$\text{Area}(S) = \iint_S 1\, dS = \iint_B |\boldsymbol{\varphi}_u(u, v) \times \boldsymbol{\varphi}_v(u, v)|\, du\, dv\ .$$

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Area of the surface $S = \varphi(B)$:

$$\text{Area}(S) = \iint_B \sqrt{1 + (f_u(u,v))^2 + (f_v(u,v))^2}\, du\, dv$$

Surfaces of Revolution

- C: parametrized curve in the $x > 0$ half plane of the $xz-$plane;
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Then

$$dS = |\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v|\, du\, dv = f(u)\sqrt{(f'(u)^2 + (g'(u))^2}\, du\, dv$$

The area of the surface is

$$\text{Area}(S) = \iint_{[a,b]\times[0,2\pi]} f(u)\sqrt{(f'(u)^2 + (g'(u))^2}\, du\, dv =$$

$$= 2\pi \int_{[a,b]} f(u)\sqrt{(f'(u)^2 + (g'(u))^2}\, du = 2\pi \int_C x\, ds$$

Pappus' First Centroid Theorem

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  - the centroid is at a distance of $\frac{2R}{\pi}$ from the axis.

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If S doesn't have a global parametrization:

- divide it into smaller pieces $S_1, \ldots S_N$ with non-overlapping interiors;
- such that each piece S_k has a local parametrization;
- then
$$\iint_S f(P)\, dS = \iint_{S_1} f(P)\, dS + \cdots + \iint_{S_1} f(P)\, dS\ .$$

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$$\iint_S z\,dS = \iint_D \sqrt{R^2 - u^2 - v^2}\,\frac{R}{\sqrt{R^2 - u^2 - v^2}}\,du\,dv = \iint_D R\,du\,dv = \pi R^3\,.$$

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The $z-$coordinate of the centroid is

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