Divergence Theorem

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Vector Fields in Space

- $\mathbf{X}$: smooth vector field defined on an open region D in space;
- p: point in D.
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  - What is the infinitesimal effect of $\mathbf{X}$ at p?
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- Two types of effects:
  - radial effect
  - rotational effect

Radial effect:

- S: a piecewise smooth surface, bounding a region B around p;
- Questions:
  - How much matter does $\mathbf{X}$ carry across S?
  - What happens when $B \to \{p\}$?

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- $\mathbf{N}$: a continuous unit vector field normal to S.
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<u>Definitions</u>:

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If the surface S bounds a domain in space:

- outward normal gives the *positive* orientation;
- inward normal gives the *negative* orientation.

Flux and Divergence

- S: an oriented surface, orientation given by the unit normal field $\mathbf{N}$;
- $\mathbf{X}$: smooth vector field on S

Definition: The flux of $\mathbf{X}$ across S is

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Radial effect of $\mathbf{X}$ at p:

- D: region around p,
- Boundary of D: $S = \partial D$, piecewise smooth parametrized surface.

<u>Definition:</u> The *divergence* of $\mathbf{X}$ at p is the density of flux at p:

$$(\operatorname{div} \mathbf{X})(p) = \lim_{D \to \{p\}} \frac{1}{\operatorname{vol}(D)} \iint_S \boldsymbol{X} \cdot \boldsymbol{N} dS\ ,$$

if the limit exists.

Computations Using Parametrizations

- $\varphi\colon D \to S$, $P = \varphi(u,v)$: smooth parametrization of S;
- $\boldsymbol{\varphi}_u$ and $\boldsymbol{\varphi}_v$ are tangent vectors;
- $\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v$ is a normal vector; may point in the direction of $\mathbf{N}$ or not;
- parametrization φ is *compatible* with the orientation given by $\mathbf{N}$ if $\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v$ and $\mathbf{N}$ point in the same direction
- Equivalently: the frame $\{\mathbf{N}, \boldsymbol{\varphi}_u, \boldsymbol{\varphi}_v\}$ is right hand oriented.
- If φ is a parametrization compatible with the orientation, then

$$\mathbf{N} = \frac{\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v}{|\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v|}$$

$$\mathbf{dS} = \mathbf{N}\, dS = \frac{\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v}{|\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v|} \cdot |\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v|\, du\, dv = \boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v\, du\, dv$$

and therefore

$$\iint_S \mathbf{X} \cdot \mathbf{N}\, dS = \iint_S \mathbf{X} \cdot \mathbf{dS} = \iint_D \mathbf{X}(\varphi(u,v)) \cdot (\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v)\, du\, dv\ .$$

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A parametrization of S is $\varphi\colon [0,\pi] \times [0, 2\pi] \to \mathbb{R}^3$,

$$\varphi(u, v) = (R \sin u \cos v, R \sin u \sin v, R \cos u) \ .$$

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Then

$$\begin{aligned}\boldsymbol{\varphi}_u =&\langle R\cos u\cos v, R\cos u\sin v, -R\sin u\rangle\\ \boldsymbol{\varphi}_v =&\langle -R\sin u\sin v, R\sin u\cos v, 0\rangle\end{aligned}$$

$$\boldsymbol{\varphi}_u\times\boldsymbol{\varphi}_v = \langle R^2\sin^2 u\cos v, R^2\sin^2 u\sin v, R^2\sin u\cos u\rangle = R\sin u\,\mathbf{r} = R^2\sin u\,\mathbf{N}$$

$$\begin{aligned}\iint_S \mathbf{X}\cdot\mathbf{dS} =& \iint_D \mathbf{X}(\varphi(u,v))\cdot(\boldsymbol{\varphi}_u\times\boldsymbol{\varphi}_v)\,du\,dv =\\ =&\int_{u=0}^{u=\pi}\int_{v=0}^{v=2\pi} a\,R\sin u\cos v\,R^2\sin^2 u\cos v\ du\,dv =\\ =&aR^3\left(\int_{u=0}^{u=\pi}\sin^3 u\,du\right)\left(\int_{v=0}^{v=2\pi}\cos^2 v\,dv\right) = cR^3\cdot\frac{4}{3}\cdot\pi = \frac{4\pi aR^3}{3}\ .\end{aligned}$$

Example: Continued

$$\iint_S ax\mathbf{i}\cdot\mathbf{dS} = \frac{4\pi aR^3}{3}, \quad \iint_S by\mathbf{j}\cdot\mathbf{dS} = \frac{4\pi bR^3}{3}, \quad \iint_S cz\mathbf{k}\cdot\mathbf{dS} = \frac{4\pi cR^3}{3}$$

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If $\mathbf{X} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, then

$$\iint_S \mathbf{X}\cdot\mathbf{dS} = \frac{4\pi R^3}{3}(a+b+c) = (a+b+c)\mathrm{vol}(B)\ ,$$

where B is the volume of the ball enclosed by the sphere S.

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If the divergence of $\mathbf{X}$ at 0 exists, then:

$$\operatorname{div}\mathbf{X}(0) = \lim_{R\to 0}\frac{3}{4\pi R^3}\iint_{S_R(0)}\mathbf{X}\cdot\mathbf{dS} = \lim_{R\to 0}(a+b+c) = a+b+c\ .$$

Another Example

Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ above the $xy-$plane, oriented upward, and $\mathbf{X} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. Compute

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Parametrization: $\varphi\colon B \to \mathbb{R}^3$, $\varphi(u,v) = (u, v, 4 - u^2 - v^2)$

$$\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v = \langle 1, 0, -2u \rangle \times \langle 0, 1, -2v \rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = 2u\,\mathbf{i} + 2v\,\mathbf{j} - \mathbf{k} \ .$$

$$\mathbf{N} = -\frac{\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v}{|\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v|} \ .$$

$$\mathbf{X} \cdot \mathbf{dS} = \mathbf{X} \cdot \mathbf{n}\, dS = \mathbf{X} \cdot \left(-\frac{\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v}{|\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v|} \right) |\boldsymbol{\varphi}_u \times \boldsymbol{\varphi}_v|\, du\, dv =$$

$$= (a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}) \cdot (-2u\,\mathbf{i} - 2v\,\mathbf{j} + \mathbf{k})\, du\, dv = (-2au - 2bv + c)\, du\, dv \ ,$$

hence

$$\iint_S \mathbf{X} \cdot \mathbf{dS} = \iint_B (-2au - 2bv + c)\, du\, dv = c \iint_B du\, dv = c \cdot 4\pi = 4\pi\, c \ .$$

Back to Divergence

Recall: The *divergence* of $\mathbf{X}$ at p is the density of flux:

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$$f(P) = \lim_{D\to\{p\}} \frac{1}{\operatorname{vol}(D)} \iiint_D f(Q)\,dV\ .$$

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However:

- the definition of $\operatorname{div}\mathbf{X}$ involves a surface integral
- the definition of average involves a triple integral

We should somehow transform the surface integral into a triple integral.

Divergence Theorem

Theorem

Let D be a compact set in space with boundary S a piecewise smooth parametrized surface, oriented by the outward normal, and let

$$\mathbf{X}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

be a smooth vector field defined on D. Then

$$\iint_S \boldsymbol{X} \cdot \boldsymbol{dS} = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

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<u>Consequence:</u>

$$(\operatorname{div}\mathbf{X})(p) = \lim_{D \to \{P\}} \frac{1}{\operatorname{vol}(D)} \iint_S \mathbf{X} \cdot \mathbf{dS} =$$

$$= \lim_{D \to \{P\}} \frac{1}{\operatorname{vol}(D)} \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,.$$

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If $\mathbf{X} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then

$$\operatorname{div}\mathbf{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \text{``}\langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle P, Q, R \rangle\text{''} = \text{``}\nabla \cdot \mathbf{X}\text{''}\ .$$

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- If $(\operatorname{div}\mathbf{X})(p) > 0$, then

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$$\iint_S \boldsymbol{X} \cdot \boldsymbol{dS} = \iiint_D \operatorname{div}\boldsymbol{X}\,dV$$

- If $(\operatorname{div}\mathbf{X})(p) > 0$, then p acts as a source;

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Example

S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the $xy-$plane, oriented *upward*; $\mathbf{X} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. Use the Divergence Theorem to compute

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- We add the disk D of radius 2 centered at the origin in the plane $z = 0$;

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The upward normal to D is $\mathbf{k}$, hence $\mathbf{X} \cdot \mathbf{dS} = \mathbf{X} \cdot \mathbf{k}\, dS = c\, dS$. Therefore

$$\iint_S \mathbf{X} \cdot \mathbf{dS} = \iint_D \mathbf{X} \cdot \mathbf{dS} = \iint_D c\, dS = c \cdot \operatorname{area}(D) = 4\pi c\, .$$

Application

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- EC: Use the Divergence Theorem to show that $\mathbf{F} = \rho V g\,\mathbf{k}$
  (V: volume of the region enclosed by S.)