Stokes' Curl Theorem

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Rotational Effect

- $\mathbf{X}$ a smooth vector field defined on an open region;
- p: a point in the region
- Effect of $\mathbf{X}$ around p:
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- Information encoded in a vector $\mathbf{Y}$
- 2D rotations are easier to understand

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Fact:

- If such a vector $\mathbf{Y}$ exists, it is unique.

If $\mathbf{A}$ and $\mathbf{B}$ are vectors and $\mathbf{A} \cdot \mathbf{u} = \mathbf{B} \cdot \mathbf{u}$ for all unit vectors $\mathbf{u}$, then $\mathbf{A} = \mathbf{B}$.

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  - we must consistently orient S;
- We will do that not just for planes, but for more general surfaces.

- S: smooth surface;
- $\mathbf{n}$: smooth unit vector field normal to S;
- $\mathbf{n}$ orients S by orienting every tangent plane to S:
  - $\mathbf{u}$, $\mathbf{v}$ are non-collinear vectors tangent to S at a point p;
  - Definition: $\{\mathbf{u}, \mathbf{v}\}$ is a positively oriented frame for the tangent plane if $\{\mathbf{n}, \mathbf{u}, \mathbf{v}\}$ is positively oriented in space.

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- S: smooth surface, oriented by unit normal vector $\mathbf{n}$;
- D: region in S, bounded by a curve $C = \partial D$;
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Recap:

- Orientation of space + unit normal to surface $\Longrightarrow$ orientation on surface
- Orientation of surface + unit normal to curve $\Longrightarrow$ orientation of curve

Example

- S: the unit sphere $x^2 + y^2 + z^2 = 1$;
- oriented by the outward normal $\mathbf{n}$;
- $D = S \cap \{z \geqslant 0\}$ be the upper hemisphere;
- Boundary of D: the circle $C = S \cap \{z = 0\}$.

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If instead we use the lower hemisphere to orient C: at $p = (1, 0, 0)$

- $\mathbf{n} = \mathbf{i}$, $\mathbf{N} = \mathbf{k}$, $\mathbf{T} = \mathbf{n} \times \mathbf{N} = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$;
- which orients C clockwise in the $\{\mathbf{i}, \mathbf{j}\}-$plane.

Components of Curl in Rectangular Coordinates

$$\mathbf{Y} \cdot \mathbf{n} = \operatorname{curl}_{\mathbf{n}}(\mathbf{orth_n}\, \mathbf{X})(p)$$

If $\mathbf{X} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then

- The $\mathbf{k}$ normal orients the $xy-$plane as $\{\mathbf{i}, \mathbf{j}\}$, because $\{\mathbf{k}, \mathbf{i}, \mathbf{j}\} \sim \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$

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is true for $\mathbf{n} = \mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Is it true for *all* $\mathbf{n}$?

Theorem (Stokes)

Let S be a smooth surface in space, oriented by the unit normal field $\mathbf{n}$. Let D be a region on S, bounded by the piecewise smooth curve $C = \partial D$, with unit tangent $\mathbf{T}$ positively oriented by the outward pointing normal $\mathbf{N}$. Let $\mathbf{X} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ be a smooth vector field on S and

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Then

$$\oint_C \mathbf{X} \cdot \mathbf{dr} = \iint_D \mathbf{Y} \cdot \mathbf{dS}\ .$$

Recall: $\mathbf{X} \cdot \mathbf{dr} = \mathbf{T}\,ds$ and $\mathbf{Y} \cdot \mathbf{dS} = \mathbf{n}\,dS$.

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Idea of proof:

- Use a parametrization of S to move the integrals to the parameter plane;

Theorem (Stokes)

Let S be a smooth surface in space, oriented by the unit normal field $\mathbf{n}$. Let D be a region on S, bounded by the piecewise smooth curve $C = \partial D$, with unit tangent $\mathbf{T}$ positively oriented by the outward pointing normal $\mathbf{N}$. Let $\mathbf{X} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ be a smooth vector field on S and

$$\mathbf{Y} = (\partial_y R - \partial_z Q)\,\mathbf{i} + (\partial_z P - \partial_x R)\,\mathbf{j} + (\partial_x Q - \partial_y P)\,\mathbf{k}\ .$$

Then

$$\oint_C \mathbf{X} \cdot \mathbf{dr} = \iint_D \mathbf{Y} \cdot \mathbf{dS}\ .$$

Recall: $\mathbf{X} \cdot \mathbf{dr} = \mathbf{T}\,ds$ and $\mathbf{Y} \cdot \mathbf{dS} = \mathbf{n}\,dS$.

Idea of proof:

- Use a parametrization of S to move the integrals to the parameter plane;
- Apply Green's Theorem in the parameter plane.

Consequences

- $\mathbf{n}$: unit vector
- S: plane through p normal to $\mathbf{n}$;
- C: simple curve in S around p;
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- $\mathbf{Y} = (\partial_y R - \partial_z Q)\,\mathbf{i} + (\partial_z P - \partial_x R)\,\mathbf{j} + (\partial_x Q - \partial_y P)\,\mathbf{k}$ works for all $\mathbf{n}$!!

Curl of a Vector Field

Definition:
The *curl* of a smooth field X is the unique vector field $\mathbf{curl}\, X$ that satisfies

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In rectangular coordinates:
If $\mathbf{X} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$, then

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Stokes' Theorem:

$$\oint_C \mathbf{X}\cdot\mathbf{dr} = \iint_D \mathbf{curl}\,\mathbf{X}\cdot\mathbf{dS}\ .$$

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$$\mathrm{div}(\mathbf{curl}\,\mathbf{X}) = \nabla \cdot (\nabla \times \mathbf{X}) = 0\ .$$

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B: ball centered at p, with boundary a sphere S centered at p.

$$\iiint_B \mathrm{div}(\mathbf{curl}\,\mathbf{X})\,dV = \iint_{S=\partial B} \mathbf{curl}\,\mathbf{X} \cdot \mathbf{dS} = \oint_{\partial S} \mathbf{X} \cdot \mathbf{dr} = 0\ ,$$

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$$\mathbf{curl}(\mathbf{grad}\,f) = \nabla \times (\nabla f) = \mathbf{0}\ .$$

D: disk centered at p, in the plane normal to $\mathbf{n}$ at p, and $C = \partial D$

$$\iint_D \mathbf{curl}\,(\mathbf{grad}\,f) \cdot \mathbf{n}\,dS = \iint_D \mathbf{curl}\,(\mathbf{grad}\,f) \cdot \mathbf{dS} = \oint_C \mathbf{grad}\,f \cdot \mathbf{dr} = 0\ ,$$

$$\mathbf{curl}\,(\mathbf{grad}\,f)(p) \cdot \mathbf{n} = \lim_{D \to \{p\}} \frac{1}{\text{area}(D)} \iint_D \mathbf{curl}(\mathbf{grad}\,f) \cdot \mathbf{n}\,dS = 0\ ;$$

since this is valid for all unit vectors $\mathbf{n}$, we conclude that $\mathbf{curl}\,(\mathbf{grad}\,f)(p) = \mathbf{0}$.