Fundamental Theorems

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May 12, 2010

A Unifying Theme

Accumulation of a quantity over the boundary of a closed domain	=	Accumulation of a derived quantity over the entire domain

- the domain is oriented;
- the orientation of the domain induces an orientation of the boundary.

Domains of Dimension One

Fundamental Theorem of Line Integrals:

- C: smooth curve, joining points A and B, oriented from A to B;
- $\partial C = \{A, B\}$: boundary of C;
- Orientation of ∂C: A, weight -1; B, weight +1;
- f a differentiable function defined on (an open neighborhood of) C.

$$f(B) - f(A) = \int_C df \quad \Longleftrightarrow \quad \int_{\partial C} f = \int_C df \ .$$

$\mathbf{r}\colon [a, b] \to \mathbb{R}^n$, smooth parametrization of C with $\mathbf{r}(a) = A$ and $\mathbf{r}(b) = B$:

$$f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = \int_C \nabla f \cdot \mathbf{dr} \ .$$

Net Change Theorem: If $f\colon [a, b] \to \mathbb{R}$ is a differentiable function, then

$$f(b) - f(a) = \int_a^b f'(x)\, dx \ .$$

Domains of Dimension Two

Stokes' Theorem:

- S: surface, oriented by unit normal field $\mathbf{n}$; D: domain on S;
- $C = \partial D$: boundary of D, oriented by the outward unit normal $\mathbf{N}$;
- $\mathbf{X}$: is a smooth field on D.

$$\oint_{\partial D} \mathbf{X} \cdot \mathbf{dr} = \iint_D \mathbf{curl X} \cdot \mathbf{dS}$$

If $\mathbf{X} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$, then

$$\mathbf{curl X} = \langle \partial_y R - \partial_z Q, \partial_z P - \partial_x R, \partial_x Q - \partial_y P \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \nabla \times \mathbf{X}\ .$$

Green's Theorem: Particular case when S is a plane, oriented by $\mathbf{k}$.

$$\oint_{\partial D} P(x, y)dx + Q(x, y)dy = \iint_D (Q_x - P_y)\, dx\, dy$$

$$\oint_{\partial D} \mathbf{X} \cdot \mathbf{dr} = \iint_D \mathrm{curl}_{\mathbf{k}} \mathbf{X}\, dA \quad , \quad \oint_{\partial D} \mathbf{X} \cdot \mathbf{n}\, ds = \iint_D \mathrm{div}\, \mathbf{X}\, dA$$

Domains of Dimension Three

Divergence Theorem:

- D, domain in $\mathbb{R}^3$;
- ∂D: boundary of D, oriented by the outward unit normal $\mathbf{n}$;
- X: smooth vector field on D.

$$\iint_{\partial D} \mathbf{X} \cdot \mathbf{dS} = \iiint_{D} \mathrm{div}\mathbf{X}\, dV \ ,$$

where $\mathbf{X} \cdot \mathbf{dS} = \mathbf{X} \cdot \mathbf{n}\, dS$.

If $\mathbf{X} = P(x, y, z)\, \mathbf{i} + Q(x, y, z)\, \mathbf{j} + R(x, y, z)\, \mathbf{k}$, then

$$\mathrm{div}\, \mathbf{X} = P_x + Q_y + R_z = \nabla \cdot \mathbf{X}.$$

$\mathbf{X} = \mathbf{grad}\, f \Longrightarrow \mathbf{curl}\, \mathbf{X} = \mathbf{curl}(\mathbf{grad}\, f) = \mathbf{0} \Longrightarrow$ condition for scalar potential

$\mathbf{X} = \mathbf{curl}\, \mathbf{G} \Longrightarrow \mathrm{div}\, \mathbf{X} = \mathrm{div}(\mathbf{curl}\, \mathbf{G}) = 0 \Longrightarrow$ condition for vector potential

Higher Dimensional Domains

Accumulation of a quantity over the boundary of a closed domain	=	Accumulation of a derived quantity over the entire domain

- the domain is oriented;
- the orientation of the domain induces an orientation of the boundary.

General Stokes Theorem:

$$\boxed{\int_{\partial M} \omega = \int_M d\omega}$$

- It would take too long to explain here what all that means,
- Will be happy to do so in a future course, *Analysis on Manifolds*.