UMass Boston, Fall 2009 — Tuesday, September 15, 2009
Math 140: Calculus I — Lecture # 5
Notes by: Catalin Zara — **Topic: Limits**

Contents

1. Limits

1.1. **Instantaneous velocity.** Object thrown vertically from height $h_0 = 1\,m$ and with initial velocity $v_0 = 10\,m/s$. Height (in m) at time t (in s):

$$h(t) = h_0 + v_0 t - \frac{1}{2} g t^2 = 1 + 10t - 5t^2$$

with the approximation $g \simeq 10 m/s^2$ for the gravitational acceleration.

Question: What is the velocity at time $t = 0.5\,s$?

Average velocity *over time interval* $[\frac{1}{2}, \frac{1}{2} + T]$:

$$\frac{\text{change in position}}{\text{change in time}} = \frac{h(\frac{1}{2}+T) - h(\frac{1}{2})}{\frac{1}{2} + T - \frac{1}{2}} = \frac{h(\frac{1}{2}+T) - h(\frac{1}{2})}{T}$$

Velocity at $t = \frac{1}{2}$ = Instantaneous velocity *at time moment* $t = \frac{1}{2}$

Estimated by average velocity over $[\frac{1}{2}, \frac{1}{2} + T]$ for T close to 0.

Problem: Can't plug in $T = 0$ (division by 0).

Solution: Look what happens around $T = 0$ but not at $T = 0$.

Algebraic computation:

$$h\left(\frac{1}{2} + T\right) - h\left(\frac{1}{2}\right) = \left(1 + 10(\frac{1}{2} + T) - 5(\frac{1}{2} + T)^2\right) - \left(1 + 10\,\frac{1}{2} - 5\,(\frac{1}{2})^2\right) = 5T - 5T^2\ .$$

Then: Average velocity on $[\frac{1}{2}, \frac{1}{2} + T]$:

$$\frac{h(\frac{1}{2}+T) - h(\frac{1}{2})}{T} = \frac{5T - 5T^2}{T} = 5 - 5T\ .$$

Question: What value do we *expect* for $5 - 5T$ as T is close to 0?
Graphically: $5 - 5T$ is close to 5 as T is close to 0.
Question: Can we keep the average velocity within 0.01 from 5?

$$\begin{aligned} 5 - 0.01 <& 5 - 5T < 5 + 0.01 \iff \\ -0.01 <& -5T < 0.01 \iff \\ -0.002 <& T < 0.002 \end{aligned}$$

Hence if we keep T within 0.002 from 0, the average velocity on $[\frac{1}{2}, \frac{1}{2} + T]$ stays within 0.01 of 5.
Can we do better that 0.01? Yes, if we keep T even closer to 0.

1.2. **Definition of Limit.** Mathematical Concept:
Limit of a function at a point.
Ingredients:

- point a;
- Function f, defined around a but not necessarily at a.

Definition: A value L is the limit of the function f at the point a (limit of $f(x)$ as x approaches a) if
we can keep the values of $f(x)$ as close to L as we want
by keeping the values of x close enough to a, but not equal to a.

Notation:

$$L = \lim_{x \to a} f(x)$$

Remarks:

- The value $f(a)$, if the function is defined at a, is irrelevant.
- If such a value L exists, it is unique (justifying *the* limit and not *a* limit.)

With the new terminology and notation:

$$\lim_{T \to 0} (5 - 5T) = 5$$

1.3. **Slope of tangent line.** Another interpretation:
Plot the graph of $h(t) = 1 + 10t - 5t^2$.
Average velocity over $[\frac{1}{2}, \frac{1}{2} + T]$:

$$\frac{h(\frac{1}{2} + T) - h(\frac{1}{2})}{\frac{1}{2} + T - \frac{1}{2}}$$

and that is the slope of the line through the points $P = (\frac{1}{2}, h(\frac{1}{2}))$ and $Q = (\frac{1}{2} + T, h(\frac{1}{2} + T))$, which is secant to the graph.

As $T \to 0$, the secant line appears to become tangent to the graph at the point P. Hence:

$$\text{Average velocity} \leftrightsquigarrow \text{Slope of secant line} \leftrightsquigarrow \frac{h(\frac{1}{2}+T)-h(\frac{1}{2})}{\frac{1}{2}+T-\frac{1}{2}}$$

$$\text{Instantaneous velocity} \leftrightsquigarrow \text{Slope of tangent line} \leftrightsquigarrow \lim_{T\to 0}\frac{h(\frac{1}{2}+T)-h(\frac{1}{2})}{\frac{1}{2}+T-\frac{1}{2}}$$

1.4. **Second example.** Heavy object attached to horizontal spring. At natural length, give an impulse. Object starts moving with initial velocity v_0.

Motion: oscillation about the initial position.

$x(t)$ = position (in cm) at time t (in s).

Measured from initial position ($x(0) = 0$) in the direction of v_0.

Possible rule for x: (from laws of physics and higher math):

$$x(t) = 2\sin(3\pi t)$$

Question: What was the initial velocity v_0?

v_0 = Instantaneous velocity at time $t = 0$

$$v_0 = \lim_{T\to 0}\frac{x(T)-x(0)}{T-0} = \lim_{T\to 0}\frac{2\sin(3\pi T)}{T}$$

Problems:

- Can't plug in $T = 0$;
- Can't use algebra to simplify.

Solution: Try to estimate the limit through numerical approximations.

$$T = 0.01 \Longrightarrow \frac{2\sin(3\pi T)}{T} \simeq 18.8217...$$
$$T = 0.001 \Longrightarrow \frac{2\sin(3\pi T)}{T} \simeq 18.8493...$$
$$T = 0.0001 \Longrightarrow \frac{2\sin(3\pi T)}{T} \simeq 18.8496...$$

It appears that as T is close to 0, the average velocity on $[0, T]$ is clusters around some value L and $L \simeq 18.849...$ It is reasonable to assume that

$$v_0 = \lim_{T\to 0}\frac{2\sin(3\pi T)}{T} \simeq 18.849...\, cm/s\ .$$

1.5. **Determining limits.**

- Graphically
- Analytically
- Numerically

Major Warning: If we know that

$$\lim_{x \to a} f(x)$$

exists, we can estimate it numerically by using values of x close to a. Otherwise, the numerical approach can produce false readings:

Example: Determine

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right).$$

Use the numerical approach:

$$x = 0.01 \Longrightarrow \sin\left(\frac{\pi}{x}\right) = \sin 100\pi = 0$$
$$x = 0.001 \Longrightarrow \sin\left(\frac{\pi}{x}\right) = \sin 1000\pi = 0$$
$$x = 0.0001 \Longrightarrow \sin\left(\frac{\pi}{x}\right) = \sin 10000\pi = 0$$

It **appears** that

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$

is likely to be 0. However, for

$$x = \frac{2}{5} \Longrightarrow \sin\left(\frac{\pi}{x}\right) = \sin\frac{5\pi}{2} = \sin\frac{\pi}{2} = 1$$
$$x = \frac{2}{2009} \Longrightarrow \sin\left(\frac{\pi}{x}\right) = \sin\frac{2009\pi}{2} = \sin\frac{\pi}{2} = 1$$
$$x = \frac{2}{200009} \Longrightarrow \sin\left(\frac{\pi}{x}\right) = \sin\frac{200009\pi}{2} = \sin\frac{\pi}{2} = 1$$

so now it **appears** that

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$

is likely to be 1.

Conclusion: if by considering different sets of choices (different ways of approaching a) you get conflicting information about the clustering value, then the limit does not exist. Hence:

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right).$$

does not exist.

Potential solution: if unsure about the existence of the limit, check using *random* values approaching a.