UMass Boston, Fall 2009
Math 140: Calculus I
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Lecture # 6
Topic: More about limits

Contents

1. More About Limits

Recall: f, function defined around a, but no necessarily at a.

$$\lim_{x \to a} f(x) = L$$

if we can keep the values of $f(x)$ as close to L as we want by keeping the values of x close enough to a, but not equal to a.

1.1. **Infinite Limits.**

$$h(x) = \frac{1}{(x-1)^2}$$

What is

$$\lim_{x \to 1} h(x) \; ?$$

(if it even exists).

Note: h is NOT DEFINED at $x = 1$. NO PROBLEM!

The question is: What happens to $h(x)$ as $x \simeq 1$ but $x \neq 1$?

Numerical approach:

$$x = 1.1 \longrightarrow h(1.1) = 100$$
$$x = 1.01 \longrightarrow h(1.01) = 10000$$
$$x = 0.999 \longrightarrow h(0.999) = 1000000$$

It appears that the closer to 1 we get, the larger the output is.

Graphical approach: We can keep the values of $h(x)$ as large as we want by keeping the values of x close enough to 1 but not equal to 1.

There is no finite number L that satisfies the condition of the limit, so one may say that the limit doesn't exist.

Enters ∞. Main ideea:

$$\text{as } \textit{large} \text{ as we want } = \text{ as close to } \infty \text{ as we want}$$

Consequently:

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty\ .$$

Similar definition for $-\infty$:

$$\text{as close to } -\infty \text{ as we want } = \text{ as } \textit{small} \text{ (negative) as we want}$$

1.2. **Limits at infinity.** Behavior of $h(x)$ as x gets larger/smaller

Numerically:

$$\text{Large } x \to \text{ large } (x-1)^2 \to \text{ close to 0 values for } h(x) = \frac{1}{(x-1)^2}$$

Graphically: We can keep $h(x)$ within 0.01 from 0 if we keep x large enough. Works for arbitrary thin horizontal bands around 0.

Analytically:

$$\left|\frac{1}{(x-1)^2}\right| < 10^{-2} \iff |x-1| > 10 \Longleftarrow x > 11$$

If we keep x above 11, then $h(x)$ is within 0.01 from 0.

Definition:

$$\lim_{x\to\infty} f(x) = L$$

if we can keep the values of $f(x)$ as close to L as we want by keeping the values of x large enough.

- L can be finite or infinite.
- Similar definition for limits at $-\infty$.

1.3. **General definition.** The definition of

$$\lim_{x\to a} f(x) = L$$

is the same for:

- finite a, L;
- finite a, infinite L;
- infinite a.

Conventions:

$$\text{close to } \infty \longleftrightarrow \text{ large}$$
$$\text{close to } -\infty \longleftrightarrow \text{ small (negative)}$$

1.4. **Side limits.** How about

$$\lim_{x\to 1} \frac{1}{x-1} \ ?$$

To the right of 1: values stay large if x stays close to 1 and $x > 1$.
To the left of 1: values stay small if x stays close to 1 and $x < 1$.

Conclusion: expected outcome depends on how we approach 1, hence

$$\lim_{x\to 1} \frac{1}{x-1} \qquad \text{DOES NOT EXIST.}$$

But: things are consistent to the right of 1, and to the left of 1.

Definition (Side limit). If f is a function defined to the right of a:

A value L is the limit of $f(x)$ as x approaches a from the right if we can keep the values of $f(x)$ as close to L as we want by keeping the values of x close enough to a and strictly greater than a.

Notation:

$$L = \lim_{x\to a^+} f(x)$$

Similar definition for

$$L = \lim_{x\to a^-} f(x)$$

Replace *greater* than a by *smaller* than a.

Remark: L can be finite or infinite.

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \qquad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty$$

Side limits at infinity:

- Same definition for

$$\lim_{x\to \infty^-} f(x) = \lim_{x\to\infty} f(x) \quad , \quad \lim_{x\to -\infty^+} f(x) = \lim_{x\to -\infty} f(x)$$

- The other side limits at infinity don't make sense.

1.5. **Limits and side limits.** If a is finite and

$$\lim_{x\to a} f(x) = L \ ,$$

then

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Consequence: If

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

(either because their values are different or because one or both don't exist), then

$$\lim_{x\to a} f(x) \qquad \text{DOESN'T EXIST!}$$

But: If

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

(both exist and their values are equal, finite or infinite), then

$$\lim_{x\to a} f(x) = L\ .$$

Summary. Possible scenarios for

$$\lim_{x\to a} f(x)\ :$$

- Exists and is finite.
- Exists, but is infinite
- Does not exist, because either:
  - The side limits are not equal, OR
  - One of the side limits doesn't even exist.

1.6. **Examples.** Consider

$$f(x) = \begin{cases} 2x-1\ , & \text{if } x<1 \\ 2\ , & \text{if } x=1 \\ x^2+2\ , & \text{if } x>1 \end{cases}$$

Question: What is

$$\lim_{x\to 1} f(x)\ ?$$

Graphically: The limit does not exist. Argumentation:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x-1) = 1$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+1) = 3$$

Since the side limits are not equal, the limit doesn't exist.

How about

$$g(x) = \begin{cases} 2x+1\ , & \text{if } x<1 \\ 2\ , & \text{if } x=1 \\ x^2+2\ , & \text{if } x>1 \end{cases}\ ?$$

Then

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (2x+1) = 3$$
$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (x^2+1) = 3$$

and therefore

$$\lim_{x\to 1} g(x) = 3\ .$$

The value of g at $x=1$ is $g(1)=2$. That value is IRRELEVANT to the computation of

$$\lim_{x\to 1} g(x)\ .$$