UMass Boston, Fall 2009
Math 140: Calculus I
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Lecture # 7
Topic: Limit Laws

Contents

1. Limit Laws

Question: If a function h is obtained from functions f and g through algebraic operations and we know

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x) \ ,$$

can we determine

$$\lim_{x\to a} h(x) \ ?$$

Good news: Yes, if the operations on limits make sense.

1.1. Limits and algebraic operations.

If the operations on limits make sense, then:

$$\begin{aligned}
\lim_{x\to a}(f(x) \pm g(x)) &= (\lim_{x\to a} f(x)) \pm (\lim_{x\to a} g(x)) \\
\lim_{x\to a}(f(x) \cdot g(x)) &= (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x)) \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \\
\lim_{x\to a} f(x)^{g(x)} &= (\lim_{x\to a} f(x))^{(\lim_{x\to a} g(x))}
\end{aligned}$$

Bad news: operations don't always make sense! Working with 0 and $\pm\infty$ can be tricky.

1.2. **Exception examples.**

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^2-4}= \tag{1}$$

$$\lim_{x\to\infty}\frac{x^2+4}{2x^3-5}= \tag{2}$$

$$\lim_{x\to\infty}\frac{x+\sqrt{2x^2+1}}{\sqrt{3x^2-1}}= \tag{3}$$

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x)= \tag{4}$$

$$\lim_{x\to 0}x^2\sin\left(\frac{\pi}{x}\right)= \tag{5}$$

$$\lim_{x\to 0}\frac{\sin(2x)}{6x}= \tag{6}$$

1.3. **Limits of rational functions at finite values.** For

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^2-4}$$

we can't apply the quotient rule because

$$\lim_{x\to 2}(x^2-3x+2)=0=\lim_{x\to 2}(x^2-4)$$

and limit of the ratio of two quantities that go to zero ("$\frac{0}{0}$") is undefined.

Solution: Factor the polynomials.

Good to know: If $P(a)=0$ then $x-a$ is a factor of P.

$$x^2-3x+2=(x-1)(x-2) \qquad x^2-4=(x-2)(x+2)$$

hence

$$\frac{x^2-3x+2}{x^2-4}=\frac{(x-1)(x-2)}{(x+2)(x-2)}=\frac{x-1}{x+2}$$

for all $x\neq 2$ and therefore

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^2-4}=\lim_{x\to 2}\frac{x-1}{x+2}=\frac{1}{4}\ .$$

1.4. **Limits of rational functions at infinity.**

$$\lim_{x\to\infty}\frac{x^2+4}{2x^3-5}$$

Can't apply the limit law for quotients, because

$$\lim_{x\to\infty}(x^2+4)=\infty \quad \text{and} \quad \lim_{x\to\infty}(2x^3-5)=\infty$$

and limit of the ratio of two quantities that go to infinity ("$\frac{\infty}{\infty}$") is undefined. Factoring won't help, either.

Solution: ORDER OF MAGNITUDE

$$
\begin{aligned}
x^2+4 =& x^2\left(1+\frac{4}{x^2}\right)\\
2x^3-5 =& x^3\left(2-\frac{5}{x^3}\right)
\end{aligned}
$$

Then

$$\lim_{x\to\infty}\frac{x^2+4}{2x^3-5}=\lim_{x\to\infty}\frac{x^2\left(1+\frac{4}{x^2}\right)}{x^3\left(2-\frac{5}{x^3}\right)}=\lim_{x\to\infty}\frac{1}{x}\frac{1+\frac{4}{x^2}}{2-\frac{5}{x^3}}$$

Since

$$\lim_{x\to\infty}\frac{4}{x^2}=0=\lim_{x\to\infty}\frac{5}{x^3}$$

we get

$$\lim_{x\to\infty}\frac{x^2+4}{2x^3-5}=\lim_{x\to\infty}\frac{1}{x}\cdot\frac{1+\frac{4}{x^2}}{2-\frac{5}{x^3}}=\lim_{x\to\infty}\frac{1}{x}\cdot\lim_{x\to\infty}\frac{1+\frac{4}{x^2}}{2-\frac{5}{x^3}}=0\cdot\frac{1}{2}=0\,.$$

General Rule: If

$$
\begin{aligned}
P(x) =& a_m x^m + a_{m-1}x^{m-1}+\cdots+a_1x+a_0\\
Q(x) =& b_n x^n + b_{n-1}x^{n-1}+\cdots+b_1x+b_0
\end{aligned}
$$

then

$$\frac{P(x)}{Q(x)}=\frac{x^m}{x^n}\cdot\frac{a_m+\frac{a_{m-1}}{x}+\cdots+\frac{a_0}{x^m}}{b_n+\frac{b_{n-1}}{x}+\cdots+\frac{b_0}{x^n}}$$

$$\lim_{x\to\infty}\frac{a_m+\frac{a_{m-1}}{x}+\cdots+\frac{a_0}{x^m}}{b_n+\frac{b_{n-1}}{x}+\cdots+\frac{b_0}{x^n}}=\frac{a_m}{b_n}\quad\text{finite}$$

$$\lim_{x\to\infty}\frac{x^m}{x^n}=\begin{cases}1,\ \text{if } m=n\\ 0,\ \text{if } m<n\\ \infty,\ \text{if } m>n\end{cases}$$

Conclusion:

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\begin{cases}\frac{a_m}{b_m}, & \text{if }\deg P=\deg Q=m\\ 0, & \text{if }\deg P<\deg Q\\ \pm\infty, & \text{if }\deg P>\deg Q\end{cases}$$

1.5. **More about order of magnitude.** Note that for $x > 0$ we have

$$x+\sqrt{2x^2+1} = x+\sqrt{x^2\left(2+\frac{1}{x^2}\right)} = x+x\sqrt{2+\frac{1}{x^2}} = x\left(1+\sqrt{2+\frac{1}{x^2}}\right)$$

and

$$\sqrt{3x^2-1} = x\sqrt{3-\frac{1}{x^2}}$$

Therefore

$$\frac{x+\sqrt{2x^2+1}}{\sqrt{3x^2-1}} = \frac{x\left(1+\sqrt{2+\frac{1}{x^2}}\right)}{x\sqrt{3-\frac{1}{x^2}}} = \frac{1+\sqrt{2+\frac{1}{x^2}}}{\sqrt{3-\frac{1}{x^2}}}$$

Then

$$\lim_{x\to\infty} \frac{x+\sqrt{2x^2+1}}{\sqrt{3x^2-1}} = \lim_{x\to\infty} \frac{1+\sqrt{2+\frac{1}{x^2}}}{\sqrt{3-\frac{1}{x^2}}} = \frac{1+\sqrt{2}}{\sqrt{3}} \,.$$

Where did we use $x > 0$? What happens for $x \to -\infty$?

1.6. **Difference Exception.** We can't apply the difference rule for

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x)$$

because

$$\lim_{x\to\infty}\sqrt{x^2+x} = \infty = \lim_{x\to\infty} x$$

and the limit of the difference of two quantities that go to infinity ("$\infty - \infty$") is undefined.

Solution: Conjugate!

$$\sqrt{x^2+x}-x = \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \frac{x}{x+\sqrt{x^2+x}}$$

Now use the order of magnitude:

$$x+\sqrt{x^2+x} = x\left(1+\sqrt{1+\frac{1}{x}}\right)$$

and therefore

$$\sqrt{x^2+x}-x = \frac{x}{x+\sqrt{x^2+x}} = \frac{1}{1+\sqrt{1+\frac{1}{x}}}$$

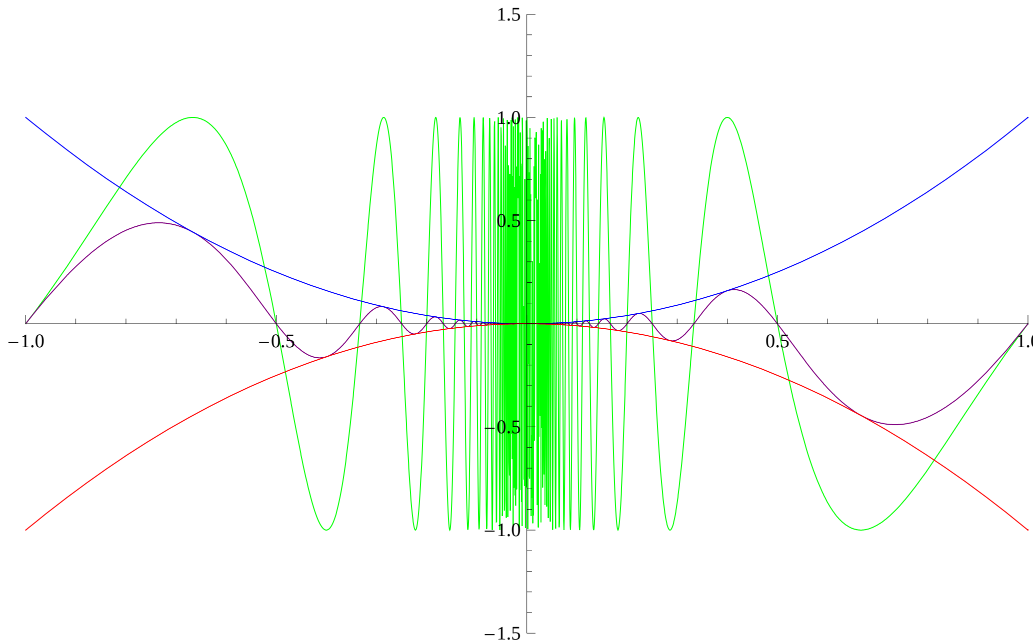
hence

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x) = \lim_{x\to\infty}\frac{1}{1+\sqrt{1+\frac{1}{x}}} = \frac{1}{2} \,.$$

1.7. **Squeeze Theorem.** We can't use the product limit law for

$$\lim_{x\to 0} x^2 \sin\frac{\pi}{x}$$

because $\lim_{x\to 0} \sin\frac{\pi}{x}$ does not exist!



Graphically: Tamed oscillations!
Mathematically: $x \to \sin\frac{\pi}{x}$ is bounded and

$$-1 \leqslant \sin\frac{\pi}{x} \leqslant 1 \Longrightarrow -x^2 \leqslant x^2 \sin\frac{\pi}{x} \leqslant x^2$$

Since both $-x^2$ and x^2 go to 0 as $x \to 0$ and $x^2 \sin\frac{\pi}{x}$ is squeezed between them, it can't escape and will also have 0 as the limit.

$$\lim_{x\to 0} x^2 \sin\frac{\pi}{x} = 0 \ .$$

General principle: SQUEEZE THEOREM.
If $f(x) \leqslant h(x) \leqslant g(x)$ for all x around a and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L \ ,$$

then

$$\lim_{x\to a} h(x) = L \ ; .$$

Consequences:

- If $\lim_{x\to a} |f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|$$

Doesn't work for non-zero limit!

- If $\lim_{x\to a} f(x) = 0$ and g is bounded around a ($|g(x)| \leqslant M$ for all x around a), then $\lim_{x\to a} f(x)g(x) = 0$.

$$-M|f(x)| \leqslant |f(x)g(x)| \leqslant M|f(x)|$$

Example:

$$\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{1}{x} \cdot \sin x = 0\ .$$

1.8. **Change of variable.** How about

$$\lim_{x\to 0} \frac{\sin x}{x}\ ?$$

It's a "$\frac{0}{0}$" exception, and we can't apply any of the methods studied so far. Important to know:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{as in} \qquad \lim_{\square\to 0} \frac{\sin \square}{\square} = 1$$

We'll use this to compute

$$\lim_{x\to 0} \frac{\sin 2x}{6x}$$

Must have $2x$ in the box. Fortunately $x \to 0$ is equivalent to $2x \to 0$ and therefore

$$\lim_{x\to 0} \frac{\sin 2x}{6x} = \lim_{2x\to 0} \frac{\sin 2x}{6x} = \lim_{2x\to 0} \frac{\sin 2x}{3\cdot 2x} = \frac{1}{3} \cdot \lim_{2x\to 0} \frac{\sin 2x}{2x} = \frac{1}{3}\ .$$