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Math 140: Calculus I — Lecture # 8
Notes by: Catalin Zara — **Topic: Continuity**

Contents

1. Continuity

1.1. **Continuous functions.** Direct Substitution Property:
If P is a polynomial function, then

$$\lim_{x\to a} P(x) = P(a)\ .$$

Definition: A function f is continuous at a point a if

$$\lim_{x\to a} f(x) = f(a)\ .$$

Intuitively: Actual value ($f(a)$) is the expected value at a (the limit).
Implicit assumption:

- For the limit to make sense $\Longrightarrow f$ is defined near a;
- For the right hand side to make sense $\Longrightarrow f$ is defined at a.

Definition: A function f is continuous if and only if it is continuous at all points where it is defined.

1.2. **Classes of continuous functions.**

- Polynomials
- Power functions
- Rational functions

  Consequence of the fact that the result of adding, multiplying, dividing, or composing functions is also a continuous function.

  Warning: Contrary to what the textbook says, the function

$$f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$$

IS continuous, because it is continuous at all points where it is defined (all nonzero real numbers).

- Trigonometric functions: sin, cos, tan (same warning: tan is not defined at $\pi/2$)

Remark: The intuitive rule

$$\text{Continuous} \iff \text{One strike graph}$$

applies ONLY if the domain is an INTERVAL.

1.3. **Discontinuities.** What can go wrong?

How can a function f fail to be continuous at a point a?

- The limit
$$\lim_{x\to a} f(x) = L$$
exists and is finite, but is not equal to $f(a)$. Two possible reasons:
  - The function is not defined at a. Then technically it doesn't make sense to talk about f being continuous or not at a. However, in this situation, f has a continuous extension
  $$g(x) = \begin{cases} f(x), & \text{if } x \neq a \\ L, & \text{if } x = a \end{cases}.$$
  - The function is defined at a, but $f(a) \neq L$. Then f has a *removable* discontinuity at a: We can eliminate the discontinuity by redefining $f(a) = L$.
- The limit does not exist or is not finite. Three possible cases:
  - The side limits
  $$\lim_{x\to a^-} f(x) \quad \text{and} \quad \lim_{x\to a^+} f(x)$$
  exist, are finite, but not equal. Then f has a *jump* discontinuity at a.
  - The side limits exist, but at least one of them is infinite. Then f has an *infinite* discontinuity at a.
  - One of the side limits does not exist. Then f has an *essential* discontinuity at a.

1.4. **Example.** Piecewise defined function:

$$f(x) = \begin{cases} x^2 + a, & \text{if } x < 1 \\ b, & \text{if } x = 1 \\ 3x - a, & \text{if } x > 1 \end{cases}$$

Question: For what values of a na db if f a continuous function?

Solution: On $(-\infty, 1)$ the function is polynomial, hence continuous at all points in $(-\infty, 1)$. Similarly, on $(1, \infty)$, the function is also polynomial, hence it is continuous at all points in $(1, \infty)$. Hence the function is continuous if and only if it is continuous at $x = 1$. By definition, this happens if

$$\lim_{x \to 1} f(x) = f(1)$$

which is equivalent to

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) \; .$$

But

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (x^2 + a) = 1 + a \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (3x - a) = 3 - a \\ f(1) &= b \end{aligned}$$

So f is continuous at 1 (and then continuous at all points) iff

$$1 + a = 3 - a = b \Longleftrightarrow a = 1 \text{ and } b = 2 \; .$$

1.5. **Intermediate Value Theorem.** True or False:

At some moment in your life you were EXACTLY one meter tall.

Intuitively, the answer is True: Set h to be the function h that gives the height $h(t)$ at time t. A reasonable value for $h(0) = 50\ cm$ (height at birth), and a reasonable value for current height $h(18) = 160\ cm$. Because growth is a continuous process, h is continuous and it can't jump from 50 to 160 without going through 100.

<u>Intermediate Value Theorem:</u> If $f \colon [a, b] \to \mathbb{R}$ is a continuous function and M is a value between $f(a)$ and $f(b)$, then there exists a point c in the interval $[a, b]$ for which $f(c) = M$.

In the height example, $f = h$, $a = 0$, $b = 18$, and $M = 100$ is a value between $h(0) = 50$ and $h(18) = 160$.

<u>Remark:</u> The IVT is an existential result. It doesn't tell us:

- How many such points c are there such that $f(c) = M$;
- How to find those points.

Role of continuity is essential. Compare to the following situation: In a game a basketball team scored 100 points. Is it guaranteed that there was a moment during the game when the team had scored exactly 50 points?

NO (it may have been, but it may also not. It is not guaranteed).

It doesn't contradict the IVT because the function that gives the score at time t is NOT continuous (has a lot of jump discontinuities!)

1.6. **Application.** Solve $x^4 = 3 - x$

Unreasonable request! Can't find exact solutions (roots). Next best thing: approximate.

Reasonable question: Find an interval of length 0.5 that contains a solution of the equation.

The equation $x^4 = 3 - x$ is equivalent to $x^4 + x - 3 = 0$. Consider $h(x) = x^4 + x - 3$. It is a continuous function, and to solve the equation is equivalent to finding c such that $h(c) = 0$.

We want to apply the IVT, with $M = 0$, and for that we need to find points a and b (at distance 0.5 from each other) such that $M = 0$ is between $h(a)$ and $h(b)$. Then one of the values $h(a)$ or $h(b)$ must be positive and the other negative. Summarizing: we look for changes in sign.

Test small values:

$$h(0) = -3 < 0$$
$$h(1) = -1 < 0$$
$$h(2) = 15 > 0$$

We found the change in sign, so for sure there is a root c between 1 and 2. Not done, because the interval is not small enough. Continue by testing the midpoint $h(3/2)$.

If $h(3/2) > 0$, then there is a change in sign from 1 to 3/2, so there must be a root in $[1, 1.5]$. If $h(3/2) < 0$, then there is a change in sign from 3/2 to 2, so there must be a root in $[1.5, 2]$. It turns out that

$$h(3/2) = \frac{81}{16} + \frac{3}{2} - 3 > 0$$

so the equation $h(x) = 0$ has a solution in the interval $[1, 1.5]$.