UMass Boston, Fall 2009
Math 140: Calculus I
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Thursday, September 24, 2009
Lecture # 9
Topic: Derivative at a Point

Contents

1. Derivative at a Point

1.1. **Motivation.** Recall what motivated us to study limits:

- Instantaneous velocity at time $t_0 = a$:

$$\lim_{T\to 0} \frac{f(a+T) - f(a)}{T}$$

- Slope of tangent line at the point $(a, f(a))$:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

In both cases, same ingredients:

- Point a,
- Function f defined at and around a.

1.2. **Definition.** If

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite, then:

- We say that the function f is *differentiable* at the point a;
- We denote the value of the limit by $f'(a)$.
- We call the **number** $f'(a)$ *the derivative of* f *at* a.

Alternative definition: Since

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \overset{x=a+h}{=\!=\!=\!=\!=\!=} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

we also have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Alternative notation:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}\bigg|_{x=a} = \frac{df}{dx}\bigg|_{x=a}$$

- Advantage: clearly identifies the independent variable
- Disadvantage: more cumbersome notation.

Interpretation:

$$f'(a) = \text{Instantaneous rate of change of } f(x)$$
$$\text{with respect to changes in } x \text{ when } x = a.$$

Applications:

- If $f(t)$ is the position at time t, then the instantaneous velocity at time $t = a$ is $f'(a)$.
- The slope of the tangent line to the graph of f at the point $(a, f(a))$ is $f'(a)$.

1.3. **Example.** For $f(x) = x^2$ and $a = 3$, compute $f'(a)$.

Solution: We need to determine whether the limit

$$\lim_{x\to 3} \frac{f(x) - f(3)}{x - 3}$$

exists and is finite. Obviously we can't use the quotient rule for limits, because $\lim_{x\to 3}(x - 3) = 0$. But for $x \neq 3$ we have:

$$\frac{f(x) - f(3)}{x - 3} = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$$

hence

$$\lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3}(x + 3) = 6\ .$$

Since the limit exists and is finite, the function f is differentiable at $a = 3$ and $f'(3) = 6$.

1.4. **Equation of tangent line.** Equation of line with slope m passing through the point $P(x_0, y_0)$:

$$y - y_0 = m(x - x_0)$$

For the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$, the slope is $m = f'(a)$. Equation:

$$y - f(a) = f'(a)(x - a) \Longrightarrow y = f(a) + f'(a)(x - a)$$

Example: Equation of line tangent to the graph of $y = \sqrt{x}$ at the point corresponding to $x = 16$.

Point: (16,4).

Slope:

$$m = y'(16) = \lim_{x\to 16} \frac{y(x) - y(16)}{x - 16} = \lim_{x\to 16} \frac{\sqrt{x} - 4}{x - 16}\,.$$

Again, we can't apply the ratio law (bottom goes to zero). But we can use the conjugate:

$$\lim_{x\to 16} \frac{\sqrt{x} - 4}{x - 16} = \lim_{x\to 16} \frac{\sqrt{x} - 4}{x - 16} \cdot \frac{\sqrt{x} + 4}{\sqrt{x} + 4} = \lim_{x\to 16} \frac{x - 16}{(x - 16)(\sqrt{x} + 4)} =$$

$$= \lim_{x\to 16} \frac{1}{\sqrt{x} + 4} = \frac{1}{8}\,.$$

The slope is $m = y'(16) = \frac{1}{8}$ and the equation of the tangent line is

$$y - 4 = \frac{1}{8}\,(x - 16) \Longleftrightarrow y = \frac{1}{8}\,x + 2\,.$$

1.5. **Points of non-differentiability.** What can go wrong?

- If f is differentiable at a then

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

  must be finite. Since $\lim_{x\to a}(x - a) = 0$, this can possibly happen only if $\lim_{x\to a}(f(x) - f(a)) = 0$. Therefore:

  f differentiable at $a \Longrightarrow f$ continuous at a.
  Equivalent statement:
  f not continuous at $a \Longrightarrow f$ not differentiable at a.

- What if f is continuous at a point? Is it necessarily differentiable at that point?

  Answer: No! It might be, but there is no guarantee..
  Example: $f(x) = |x|$ is continuous at 0, but:

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

  and

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{x}{x} = 1$$

  Hence the side-limits exist, are finite, but not equal. Therefore f is not differentiable at 0. Such a point is called a *corner point*.

- It may also be possible that the limit in the definition of differentiability exists, but is infinite. Example: $f(x) = \sqrt[3]{x}$ at $x = 0$. Then we say that we have a *vertical tangent*.

- Another alternative is that one side limit in the definition of differentiability is finite and the other is infinite. Then we still have a corner point.
- If one of the side limits is $-\infty$ and the other one is ∞, then we have a *cusp point*.

Remark: Counter-intuitive as it may be, the tangent line at a point may actually cut the graph! Think $f(x) = x^3$ and $x = 0$. The slope of the tangent line turns out to be 0, and the tangent line is the horizontal line $y = 0$. It crosses the graph at the point of tangency!