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Math 140: Calculus I — Lecture # 10
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1. Derivative as a Function

1.1. **Example.** Consider the function $g(x) = x^3$. To determine the equation of the tangent line at a point:

- Compute $g'(a)$ for every particular point a where we want the tangent line, OR
- Compute $g'(a)$ for a generic a and than plug-in the value of a for every particular point we are interested in.

Using the second approach:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^3 - a^3}{x - a} = \lim_{x\to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} =$$
$$= \lim_{x\to a}(x^2 + ax + a^2) = 3a^2$$

Conclusion: the limit exists and is finite for all a.
We produced a rule for computing $f'(a)$ for generic a: $f'(a) = 3a^2$.

1.2. **Definition.** Derivative function:
Given a function f, the function g that assigns to every point x where f is differentiable the value $g(a) = f'(a)$.

- g is customarily denoted by f' or $f^{(1)}$.
- The domain of the derivative function f' is the set of points where f is differentiable.
- The domain of the derivative function f' is included in the domain of the original function;

- The domain of the derivative function may be different than the domain of the function. Example: $f(x) = |x|$. Then

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

The domain of f is $\mathbb{R}$, and the domain of f' is $\mathbb{R} \setminus \{0\}$.

Notation:

$$f' = \frac{df}{dx} \qquad \text{Also:} \qquad (x^3)' = 3x^2 \quad \text{and} \quad \frac{d(x^3)}{dx} = 3x^2 \,.$$

Remark:

- $f'(a)$: derivative of f at a: NUMBER
- f': derivative of f: FUNCTION

1.3. **Higher derivatives.** Consider $f(x) = \frac{1}{x}$, defined on $(0, \infty)$. To compute the derivative $f'(x)$ we have two options:

- Compute $f'(a)$ and then replace a with x, or
- Compute $f'(x)$ directly as

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

Using the second method:

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x} \frac{\frac{1}{y} - \frac{1}{x}}{y - x} = \lim_{y \to x} \frac{\frac{x-y}{xy}}{y - x} = \lim_{y \to x} \frac{x - y}{xy(y - x)} = \lim_{y \to x} \frac{-1}{xy} = -\frac{1}{x^2}$$

Hence

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

How about the derivative of the derivative:

$$\left(-\frac{1}{x^2}\right)' = \lim_{y \to x} \frac{(-\frac{1}{y^2}) - (-\frac{1}{x^2})}{y - x} = \lim_{y \to x} \frac{y^2 - x^2}{x^2 y^2 (y - x)} = \lim_{y \to x} \frac{y + x}{x^2 y^2} = \frac{2x}{x^4} = \frac{2}{x^3}$$

So:

$$\text{Function } f \overset{D}{\Longrightarrow} \text{Derivative function } f' \overset{D}{\Longrightarrow} \text{Derivative of } f' = (f')' = f''$$

- f'': second order derivative of f. Also denoted by $f^{(2)}$
- $f''' = (f'')'$: third order derivative of f. Also denoted by $f^{(3)}$.
- In general: $n^{th}-$order derivative

$$f^{(n)} = (f^{(n-1)})'$$

Example:

$$\left(\frac{1}{x}\right)'' = \frac{2}{x^3}$$

1.4. **Graph of derivative function.** How to get the graph of f' from the graph of f.

Examples:

- $f(x) = \frac{1}{x}$;
- $f(x)$ general cubic.

1.5. **Piecewise functions.** Going back to piecewise functions.

For what values of a and b if the function

$$f(x) = \begin{cases} x^2 + a, & \text{if } x < 1 \\ b, & \text{if } x = 1 \\ 3x - a, & \text{if } x > 1 \end{cases}$$

differentiable at $x = 1$?

Solution: To be differentiable, the function must be continuous at $x = 1$. From a previous class meeting, that happens only if $a = 1$ and $b = 2$. So the question really is: Is the function

$$f(x) = \begin{cases} x^2 + 1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ 3x - 1, & \text{if } x > 1 \end{cases}$$

differentiable at $x = 1$?

We need to determine whether

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

exists and is finite. For that we compute the side limits:

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 + 1 - 2}{x - 1} = \lim_{x \to 1^-} (x + 1) = 2$$

and

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{3x - 1 - 2}{x - 1} = \lim_{x \to 1^+} 3 = 3$$

The side limits are not equal, therefore the limit does not exist, which means that there for $a = 1$ and $b = 2$, the function is not differentiable at $x = 1$. As those are the only values of a and b for which the function is continuous, it follows that there are no values of a and b for which the function is differentiable at $x = 1$.