UMass Boston, Fall 2009 — Thursday, October 1, 2009
Math 140: Calculus I — Lecture # 13
Notes by: Catalin Zara — **Topic: Differentiation Formulas**

Contents

1. Differentiation Formulas

1.1. **General approach.** Recall definition of derivative at a point:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\,.$$

A: Use the limit definition to compute derivatives of simple functions;

B: Develop rules that show how derivatives interact with operations on functions.

General approach for computing derivatives:

- Given a function, decompose it as a
  - sum,
  - difference,
  - product,
  - quotient,
  - composition

  of simple functions.
- Use the rules from B to get the derivative of the original function from the derivatives of the blocks.
- Use results of A to find the derivatives of the simple blocks.

1.2. **Power functions.**

1.2.1. *Constant functions.* If $f(x) = c$ for all x, then $f'(x) = 0$.

$$\boxed{(c)' = \frac{dc}{dx} = 0}$$

1.2.2. *Powers.* We have already computed derivatives of some power functions:

$$\begin{aligned}(x^2)' =& 2x\\ (x^3)' =& 3x^2\\ \left(\frac{1}{x}\right)' = (x^{-1})' =& -x^{-2} = -\frac{1}{x^2}\\ (x^{-2})' =& -2x^{-3}\\ (\sqrt{x})' = (x^{1/2})' =& \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}\end{aligned}$$

The Power Rule:

$$\boxed{(x^a)' = \frac{d\,x^a}{dx} = a\,x^{a-1}}$$

Works for a natural, integer, rational, real!

Warning: Do not apply the Power Rule for EXPONENTIAL functions!

$\dfrac{dx^\pi}{dx} = \pi x^{\pi-1}$ BUT $\dfrac{d\pi^x}{dx}$ can't be computed using this rule!

More examples:

$$\begin{aligned}f(x) = \sqrt[3]{x} = x^{1/3} \Longrightarrow& \frac{df}{dx} = \frac{1}{3}x^{-2/3}\\ g(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \Longrightarrow& \frac{dg}{dx} = -\frac{1}{2}x^{-3/2}\\ h(x) = t^2 \Longrightarrow& \frac{dh}{dx} = 0 \quad \text{(constant function, not power!)}\end{aligned}$$

1.3. **Sum, difference, constant multiple rules.** f and g are functions and c is a constant.

Sum/Difference rule:

$$\boxed{(f \pm g)' = f' \pm g'}$$

The derivative of the sum is the sum of the derivatives.
The derivative of the difference is the difference of the derivatives.

Constant multiple rule:

$$\boxed{(cf)' = cf' \text{ if } c \text{ is a constant.}}$$

Example: $P(x) = 2x^3 - 7x + 2$, polynomial function.

$$\begin{aligned}P'(x) =& (2x^3 - 7x + 2)' = (2x^3)' - (7x)' + (2') = \\ =& 2(x^3)' - 7(x)' + (2)' = \\ =& 2 \cdot 3x^2 - 7 \cdot 1 + 0 = \\ =& 6x^2 - 7\end{aligned}$$

Another example: $f(x) = 3x^4 - 2\sqrt[3]{x^2} = 3x^4 - 2x^{2/3}$. Then

$$f'(x) = 3 \cdot 4x^3 - 2 \cdot \frac{2}{3}x^{-1/3} = 12x^3 - \frac{4}{3}\,x^{-1/3}$$

How about $g(x) = x^2\sqrt{x}$? Looks like a product, but $g(x) = x^{5/2}$, so

$$g'(x) = \frac{5}{2}\,x^{3/2}\ .$$

1.4. **Product Rule.** Example: The length of a rectangle starts at 5 cm and grows at a rate of 2 cm/s. The width starts at 12 cm and grows at a rate of 3 cm/s. How fast is the area growing?

$$L(t) = 5 + 2t \qquad \text{and} \qquad W(t) = 12 + 3t$$

Then

$$A(t) = L(t) \cdot W(t) = (5 + 2t)(12 + 3t) = 5 \cdot 12 + 5 \cdot 3t + 2t \cdot 12 + 2t \cdot 3t$$

hence

$$A(t) = 5 \cdot 12 + 5 \cdot 3t + 2t \cdot 12 + 2 \cdot 3t^2$$

Therefore

$$A'(t) = (5 \cdot 12 + 5 \cdot 3t + 2t \cdot 12 + 2 \cdot 3t^2)' = 5 \cdot 3 + 2 \cdot 12 + 2 \cdot 3 \cdot 2t$$

and when $t = 0$ we get:

$$A'(0) = 5 \cdot 3 + 2 \cdot 12\ .$$

What do these numbers represent?

$$5 = L(0) \quad 3 = W'(0) \quad 2 = L'(0) \quad 12 = W(0)$$

Conclusion:

$$(L \cdot W)'(0) = L(0) \cdot W'(0) + L'(0) \cdot W(0)\ .$$

Product rule:

$$\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x)}$$

Derivative of product = derivative of first factor times the second factor plus first factor times derivative of the second factor.

Example:

$$(x^2\sqrt{x})' = (x^2)'\sqrt{x} + x^2(\sqrt{x})' = 2x\sqrt{x} + x^2 \cdot \frac{1}{2}\, x^{-1/2} = \frac{5}{2}\, x^{3/2}$$

1.5. **Quotient rule.** We know how to differentiate polynomials and product, but how about rational functions, which are quotients of polynomials?

Let $f(x) = \frac{P(x)}{Q(x)}$. Then $P(x) = f(x)Q(x)$, hence

$$P'(x) = (fQ)'(x) = f'(x)Q(x) + f(x)Q'(x)$$

Solving for $f'(x)$ we get

$$f'(x) = \frac{P'(x) - f(x)Q(x)}{Q(x)} = \frac{P'(x) - \frac{P(x)}{Q(x)}Q'(x)}{Q(x)} = \frac{P'(x)Q(x) - P(x)Q'(x)}{Q^2(x)}$$

We never used that P and Q are polynomials. Works in general:

Quotient Rule:

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}}$$

Example:

$$g(x) = \frac{1}{x^2 - 1}$$

$$g'(x) = \frac{1' \cdot (x^2 - 1) - 1 \cdot (x^2 - 1)'}{x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2}$$

Second derivative:

$$g''(x) = (g')'(x) = \left(\frac{-2x}{(x^2 - 1)^2}\right)' = \frac{(-2x)'(x^2 - 1)^2 - (-2x)[(x^2 - 1)^2]'}{[(x^2 - 1)^2]^2}$$

How do we compute the derivative $[(x^2 - 1)^2]'$? By expanding the power:

$$[(x^2 - 1)^2]' = (x^4 - 2x^2 + 1)' = 4x^3 - 4x = 4x(x^2 - 1)\ .$$

Therefore:

$$\begin{aligned} g''(x) =& \frac{-2(x^2 - 1)^2 + 2x \cdot 4x(x^2 - 1)}{(x^2 - 1)^4} = \\ =& \frac{(x^2 - 1)(-2(x^2 - 1) + 8x^2)}{(x^2 - 1)^4} = \frac{6x^2 + 2}{(x^2 - 1)^3} \end{aligned}$$

If we want to compute higher order derivatives, we will have to differentiate $(x^2 - 1)^n$ for bigger values of n, and expanding the polynomial will very soon become quite laborious. We will learn another method for computing such derivatives.